Explaining the Volatility Surface: A Closed-Form Solution to Option Pricing in a Fractional Jump-Diffusion Market

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Explaining the volatility surface: A closed-form solution to option pricing in a Fractional Jump-Diffusion Market

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Abstract

This paper prices European options in a framework that captures both non-normality of returns and serial correlation within financial time series. The underlying security dynamics are driven by a jump-diffusion process where the diffusion part is fractional Brownian motion while jumps exhibit a double-exponential distribution. These model characteristics suffice to overcome most of the evident drawbacks of the classical Black-Scholes setting, while the parsimony of my model still ensures analytical tractability.

Due to market incompleteness, I suggest an equilibrium model à la Brennan (1979). I derive a closed-from solution to the problem, which contains the Black-Scholes pricing formulae and the formulae of Kou (2002) as limit cases.

As an intuitive illustration of the model’s power, I choose the phenomenon of volatility surfaces: I show that the derived formulae are able to reflect observable patterns of real market data as the model entails a smile over moneyness as well as a non-flat term structure of implied Black–Scholes volatilities.

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I Introduction

It is well-known that the option prices derived within the classical Brownian motion framework of Black and Scholes (1973) and Merton (1973) do not describe appropriately the prices that can be observed in real financial markets. Most easily, this kind of deviation can be ascertained by extracting implied volatilities from market prices (see Derman and Kani (1994)). While the Black-Scholes pricing model would suggest that implied volatility should be a constant irrespective of the moneyness and time to maturity of the option contract, one typically receives smile or smirk type curves of volatility over moneyness as well as a non-flat term-structure of volatility.

From a modeling perspective, at least two reasons for these drawbacks of the classical model can be identified. First, pure Brownian motion models can only capture normally distributed returns. Second, by nature, Brownian motion dispenses with serial correlation of returns.

A magnitude of models have been proposed during the last decades to compensate for these shortcomings, mainly focussing on the distributional aspects of returns. Lévy processes have proven to be a powerful modeling tool allowing to depict nearly any distribution one might think of (for an overview, see e.g. Cont and Tankov (2004)). However, very few of these models can be solved analytically which might be the main reason why the Black-Scholes pricing formula is still popular amongst practitioners. One positive exception is given by Kou (2002) who managed to derive a closed-form solution within a framework that combines classical Brownian motion with a double-exponential jump process.
On the other hand, there are only few continuous-time models that include serial correlation of returns. The most promising approaches use fractional Brownian motion instead of classical Brownian motion as diffusion process which was originally introduced by Mandelbrot and van Ness (1968). During the last years, however, option pricing with respect to fractional Brownian motion was stuck. An intense debate within the field of mathematical finance had preceded discussing the fact whether financial models with fractional Brownian motion are sensible at all. Amongst others, Rogers (1997) and Sottinen (2001) derived arbitrage possibilities in specific scenarios and stated that such a model was an absurd candidate for modeling stock prices. However, Cheridito (2003) showed that these arbitrage possibilities disappear as soon as investors are exposed to an arbitrary small time interval lying between two consecutive transactions and are thereby restricted to discontinuous trading strategies. Most interestingly, Bayraktar et al. (2006) take a market microstructure perspective and show that it is exactly this discontinuity of trading which they call inertia that is one possible source to create the persistent fractional Brownian character within the price process.

In this paper, I combine the desirable properties of non-normality and serial correlation of returns within one model. While there seems to be a tradeoff between more realistic models on the one hand and analytical tractability on the other hand, the main advantage of this model is that it still can be solved analytically. To the best of the author’s knowledge, this is the first approach to combine the two desirable properties of distributions with excess kurtosis and a price process with serial correlation.

For the price process of the underlying, I suggest a jump-diffusion where the diffusion
part is driven by fractional Brownian motion and the jump part is modeled by double-
exponential distribution. Figure 1 shows an example for trajectories of such a stochastic
process. The three trajectories are identical with respect to their jump part but different
with respect to their diffusion component. Typically, the fractional part can exhibit three
different patterns that are fully described by only one crucial parameter, the so called
Hurst parameter $H$. This parameter lies between zero and one and we distinguish the
following cases: For $H = \frac{1}{2}$, fractional Brownian motion is identical to the familiar case of
classical Brownian motion. This is convenient as it allows for quick double-checking – in
this special case, all results I receive should be compatible to results received within prior
non-fractional models. In the case that $H < \frac{1}{2}$, usually called anti-persistence, paths
show kind of a mean-reverting character. This leads – in comparison to the classical
case – to stronger fluctuations in the short run, but smaller deviations from the mean
in the long run. For the empirically most relevant case of positive serial correlation or
persistence, i.e. when $H > \frac{1}{2}$, the trend behavior of the diffusion part is reinforced and
the paths show longer cycles than for classical Brownian motion. For a detailed survey
of time series properties of fractional Brownian motion, see for example Doukhan et al.
(2003).

Like the model of Kou (2002), the fractional jump-diffusion model is dynamically incom-
plete by the nature of the jumps, which along the way eliminates the arbitrage problems of
fractional Brownian motion mentioned above. With respect to methodology, this means
that it is not longer possible to price derivatives by applying continuous hedging argu-
ments. Instead, one is left to the economic concept of equilibrium valuation with respect
Figure 1: Paths of fractional jump-diffusions for different Hurst parameters. The green line is an anti-persistent trajectory (H=0.25), the red line is associated with independent increments (H=0.5) and the blue line is the path of a persistent jump-diffusion (H=0.75).

To discrete trading points.

From an economic perspective, this does not need to be regarded as a conceptual drawback: Brennan (1979) lists a number of reasons for which types of models the assumption of discrete trading points is superior, e.g. tax liabilities, real options on non-traded assets, etc. For this reason, he calls the discrete time approach to be "the major competing paradigm to the continuous time framework". Most regrettably, this insight sometimes seems to be buried in oblivion. In this article, I recollect this knowledge which enables us to solve the puzzle of option pricing in markets driven by fractional Brownian motion and double-exponential jumps.

While there are a number of similarities to the framework of Kou (2002), the mathematics of this article show one crucial distinction. As I assume the diffusion part to be generated by geometric fractional Brownian motions, the persistence of these processes necessitates
to include the price histories up to present into investors’ consideration. Consequently, all end-of-period distributions are conditional on this historic information. The technical implementation of this circumstance is a major task: Based on given results on the conditional distribution of arithmetic fractional Brownian motion, a conditional version of a multidimensional fractional Itô theorem needs to be developed. Besides technical innovation, the economic enhancements are of much higher interest: The introduction of serial correlation implies a non-linear relationship between maturity and variance of a stock price distribution. With respect to option prices, the serial correlation of the underlying can be identified as a separate source of a non-flat term structure of implied volatilities.

The rest of the paper is structured as follows: In the following section, I introduce the framework as an extension of the approach of Brennan (1979) and Kou (2002), respectively. In section 3, the basic equilibrium condition is exploited and the stochastic discount factor is derived. In section 4, the central result is provided: a closed-form solution for European options.

II Market setup and pricing equilibrium

The market setting consists of a riskless bond \( S_0 \), as well as a finite number of assets \( S_i \) \((i = 1, \ldots, M)\), whose differential equations are given by

\[
\begin{align*}
    dS_0(t) &= rS_0(t) dt, \\
    dS_i(t) &= \mu_i S_i(t) dt + \sum_{j=1}^{J} \sigma_{ij} S_i(t) dB^H_{j}(t) + S_i(t) d \left( \sum_{k=1}^{\mathcal{N}(t)} \theta_i (V_k - 1) \right)
\end{align*}
\]
The diffusion parts are driven by a finite number $J \in \mathbb{N}$ of fractional Brownian motions. The Hurst parameters $H_j$ are constant over time and not necessarily different. The diffusions can be grouped into three characteristic classes: For $H_j = \frac{1}{2}$, processes coincide with standard Brownian motion. On the other hand, Hurst parameters $H_j \in (0, \frac{1}{2})$ describe anti-persistent behavior, while processes with $H_j \in (\frac{1}{2}, 1)$ imply persistent behavior. The coefficients $r, \mu_i$ and $\sigma_{ij}$ are assumed to be constants symbolizing the riskless interest rate, the drift rates of the stocks and their volatilities with respect to the according diffusions. The fractional Brownian stochastic differentials are interpreted using a generalized Itô type integration concept which is called Wick-Itô calculus\textsuperscript{1}.

The jump parts are modeled as in Kou (2002) by the combination of a Poisson process $N(t)$ with arrival rate $\lambda$ and a series of independent identically distributed random variables $V_k$ where $Y = \ln(V)$ is double exponentially distributed with density

\[ f_Y(y) = p \cdot \eta_1 e^{-\eta_1 y} \mathbb{1}_{y \geq 0} + q \cdot \eta_2 e^{\eta_2 y} \mathbb{1}_{y < 0}, \quad \eta_1 > 0, \quad \eta_2 > 0. \tag{3} \]

That is, with probability $p$, one observes an upward jump that is exponentially distributed with mean $1/\eta_1$ and with probability $q = 1 - p$, the jump is oriented downwards with mean $1/\eta_2$.

To keep calculations as simple as possible, all stochastic processes $B^{H_j}(t), N(t)$ and $Y$ are assumed to be independent.

\textsuperscript{1}For an introduction to this integration concept with respect to fractional Brownian motion as well as its advantages over pathwise integration, see e.g. Duncan et al. (2000).
The explicit representations of the future random values are given as:

\[ S_i(T) = S_i(t) \exp \left( \mu_i(T-t) - \frac{1}{2} \sum_{j=1}^{J} \sigma_{ij}^2 (T-t)^{2H_j} \right) \]

\[ + \sum_{j=1}^{J} \sigma_{ij} \left( B_j^{H_j}(T) - B_j^{H_j}(t) \right) \prod_{k=N(t)+1}^{N(T)} \theta_k V_k. \]  

(4)

(5)

For an investor with a portfolio of \( n_0 \) units of money invested in the riskless bond as well as \( n_i \) units of money invested in the risky assets \( S_i \), the wealth \( W \) of the investor is described by

\[ W = \sum_{i=0}^{M} n_i \cdot S_i. \]  

(6)

For constant amounts \( n_i \), one can write

\[ dW(t) = \sum_{i=0}^{M} n_i \cdot dS_i(t) \]

\[ = n_0 r S_0(t) dt + \sum_{i=1}^{M} n_i \mu_i S_i(t) dt \]

\[ + \sum_{i=1}^{M} n_i \left( \sum_{j=1}^{J} \sigma_{ij} S_i(t) dB_j^{H_j}(t) + S_i(t) d \left( \sum_{k=1}^{N(t)} \theta_k (V_k - 1) \right) \right). \]  

(7)

In case that the wealth process \( W \) is bounded away from zero, it can be represented as follows:

\[ dW(t) = \omega_0 r W(t) dt + \sum_{i=1}^{M} \omega_i \mu_i W(t) dt \]

\[ + \sum_{i=1}^{M} \omega_i \left( \sum_{j=1}^{J} \sigma_{ij} W(t) dB_j^{H_j}(t) + W(t) d \left( \sum_{k=1}^{N(t)} \theta_k (V_k - 1) \right) \right), \]  

(8)

where \( \omega_i = \frac{n_i S_i}{W} \) are the percentage portfolio holdings.
Summarizing one receives

\[ dW(t) = \mu_W W(t) \, dt + \sum_{j=1}^{J} \sigma_{W_j} W(t) \, dB_j^H(t) + W(t) \, d \left( \sum_{k=1}^{N(t)} \theta_W (V_k - 1) \right) \] (9)

where

\[ \mu_W = \omega_0 r + \sum_{i=1}^{M} \omega_i \mu_i \] (10)

and

\[ \sigma_{W_j} = \sum_{i=1}^{M} \omega_i \sigma_{ij} \] (11)

and

\[ \theta_W = \sum_{i=1}^{M} \omega_i \theta_i \] (12)

Now, I consider the single period decision problem where the representative investor maximizes his utility by choosing today’s consumption level and the number of units invested in each of the \( N \) risky securities, i.e.

\[
\max_{\{c(t); n_i\}} \left[ U(c(t)) + E_t \left[ V \left( (W(t) - c(t)) e^{r(T-t)} + \sum_{i=1}^{N} n_i \left( S_i(T) - S_i(t) e^{r(T-t)} \right) \right) \right] \right] ,
\] (13)

where \( c(t) \) is the initial consumption of the representative investor, \( U(\cdot) \) is the utility function defined over initial consumption, \( V(\cdot) \) is the utility function defined over end of period wealth \( W(T) \) and \( E_t [\cdot] \) denotes the operator of conditional expectation based on the \( \sigma \)-algebra \( \mathcal{F}_t \), where the latter represents the information available at time \( t \). Note that the subscript \( t \) is of major relevance due to the non-Markovian character of fractional Brownian motion.

The first order conditions related to this decision problem are:

\[
U'(c(t)) - e^{r(T-t)} E_t \left[ V'(W(T)) \right] = 0 \] (14)

\[
E_t \left[ V'(W(T)) S_i(T) \right] - S_i(t) e^{r(T-t)} E_t \left[ V'(W(T)) \right] = 0 \ \forall i \] (15)
Consequently, the initial price of a specific asset $S_i$ is given by:

$$S_i(t) = e^{-r(T-t)} \frac{E_t[V'(W(T))S_i(T)]}{E_t[V'(W(T))]}$$

$$= e^{-r(T-t)} E_t \left[ \frac{V'(W(T))}{E_t[V'(W(T))]}S_i(T) \right]$$

$$= e^{-r(T-t)} E_t \left[ z(t,T)S_i(T) \right], \quad (16)$$

where the stochastic discount factor $z(t, T)$ is given by

$$z(t, T) = \frac{V'(W(T))}{E_t[V'(W(T))]} \quad . \quad (17)$$

Obviously, the stochastic discount factor equals the conditional expected relative marginal utility of wealth. Equation (16) represents our pricing equilibrium condition and will be used later on in order to determine the drift parameter of the underlying asset.

Analogously, the price of any contingent claim $C_{S_i}$ whose payoff in time $T$ depends solely on $S_i(T)$ can be written as:

$$C_{S_i}(t) = e^{-r(T-t)} E_t \left[ z(t,T)C_{S_i}(T) \right]. \quad (18)$$

By means of equation (18), it will in section IV be possible to calculate the price of a European call option.

### III The stochastic discount factor

In the following section, I derive the stochastic discount factor to be applied for the valuation of any contingent claim written on the asset $S$.

With respect to their end of period wealth, investors are assumed to show constant relative risk aversion described by a power-utility function, i.e.

$$V(W) = \frac{1}{1-\gamma} W^{1-\gamma}, \quad (19)$$
where $\gamma$ is the parameter of risk aversion.

Comparing the setup of our decision problem in the previous section with that of Kou (2002), the main difference is the conditionality of expectations on all available historic information. But – as motivated in the introduction – this information is highly relevant within the fractional setting and hence heavily influences asset prices. In the appendix, I therefore provide a useful result with respect to the conditional distribution of $B^H(T)$.

Furthermore, I use a conditional extension of the multi-dimensional fractional Itô theorem provided by Biagini and Øksendal (2003) to derive the value of wealth $W$ at time $T$ conditional on all information available at current time $t$, denoted by $\mathcal{F}_t$:

$$
W(T)|_{\mathcal{F}_t} = W(t) \cdot \exp \left[ \frac{1}{2} \sum_{j=1}^{M} \nu_{H_j} \sigma_{W_j}^2 (T-t)^{2H_j} \right. \\
\left. + \sum_{j=1}^{M} \sigma_{W_j} \left( B^H_{j}(T) - B^H_{j}(t) \right) \right] \prod_{k=N(t)+1}^{N(T)} \theta_{WV_k}, \tag{20}
$$

where the $\nu_{H_j}$ are defined as follows:

$$
\nu_{H_j} = \begin{cases} 
\frac{\sin(\pi(H_j - \frac{1}{2}))}{\pi(H_j - \frac{1}{2})} \frac{\Gamma(\frac{3}{2} - H_j)^2}{\Gamma(2 - 2H_j)} & \text{for } H_j \neq \frac{1}{2} \\
1 & \text{for } H_j = \frac{1}{2}
\end{cases}. \tag{21}
$$

Depending on the Hurst parameter, the factor $\nu_{H_j}$ takes the values that can be seen in Figure 2. It plots the variance reduction of conditional fractional Brownian motion based on available information about the past. For $H = \frac{1}{2}$, the factor reaches its maximum which is one. In this case, there is no difference between unconditional and conditional variance, as history plays no role in the classical Brownian motion based setting.

The proof of equation (20) is rather technical and available upon request from the author.

It follows that the marginal utility of wealth $V'(W(T)) = W(T)^{-\gamma}$ has the following
conditional representation:

\[
V'(W(T))|_{\mathfrak{F}_t} = W(t)^{-\gamma} \cdot \exp \left( -\gamma \mu_W (T - t) + \frac{1}{2} \sum_{j=1}^{J} \nu_{H_j} \sigma_{W_j}^2 (T - t)^{2H_j} \right. \\
\left. -\gamma \sum_{j=1}^{J} \sigma_{W_j} \left( B_{H_j}^j(T) - B_{H_j}^j(t) \right) \right) N(T) \prod_{k=N(t)+1}^{N} (\theta_W V_k)^{-\gamma}
\] (22)

Due to independence of the diffusion parts and the jump process, the conditional expectation of this marginal utility of wealth is then

\[
E_t \left[ V'(W(T)) \right] = W(t)^{-\gamma} \cdot \exp \left( -\gamma \mu_W (T - t) + \frac{1}{2} \sum_{j=1}^{J} \nu_{H_j} \sigma_{W_j}^2 (T - t)^{2H_j} \right) \left. \right. \\
\left. \times E_t \left[ \exp \left( -\gamma \sum_{j=1}^{J} \sigma_{W_j} \left( B_{H_j}^j(T) - B_{H_j}^j(t) \right) \right) \right] \cdot E_t \left[ \prod_{k=N(t)+1}^{N} (\theta_W V_k)^{-\gamma} \right] \right. \\
= W(t)^{-\gamma} \cdot \exp \left( -\gamma \mu_W (T - t) + \frac{1}{2} \gamma (\gamma + 1) \sum_{j=1}^{J} \nu_{H_j} \sigma_{W_j}^2 (T - t)^{2H_j} \right. \\
\left. -\gamma \sum_{j=1}^{J} \sigma_{W_j} \mu_{H_j}^j \left( T, t \right) + \lambda \zeta_t^{-\gamma} (T - t) \right), \quad (23)
\]

where \( \zeta_t^a = E \left[ V^a - 1 \right] \) as defined in Kou (2002).
Inserting (22) and (23) into (17), one obtains the stochastic discount factor

\[
 z(t, T) = \exp \left( -\gamma \sum_{j=1}^{J} \sigma_{W_j} \left( B_{j}^{H_j}(T) - B_{j}^{H_j}(t) \right) + \gamma \sum_{j=1}^{J} \sigma_{W_j} \mu_{T,t}^{H_j} \right) 
 \]

\[
 -\frac{1}{2} \gamma^2 \sum_{j=1}^{J} \nu_{H_j} \sigma_{W_j}^2 (T-t)^{2H_j} - \lambda \zeta \gamma (T-t) \right) \prod_{k=N(t)+1}^{N(T)} \left( \theta W_{V_k} \right)^{-\gamma}
\]

This stochastic discount factor defines a change of measure from \( P \) to \( P^* \): For any \( j \), the random variable \( \bar{B}_{j}^{H_j}(T) = B_{j}^{H_j}(T) + \gamma \sigma_{W_j} \nu_{H_j} (T-t)^{2H_j} - \mu_{T,t}^{H_j} \) has a conditional distribution that is normal and has the conditional \( P \)-moments

\[
 E_t \left[ \bar{B}_{j}^{H_j}(T) \right] = B_{j}^{H_j}(t) + \gamma \sigma_{W_j} \nu_{H_j} (T-t)^{2H_j}
\]

\[
 Var_t \left[ \bar{B}_{j}^{H_j}(T) \right] = \nu_{H_j} (T-t)^{2H_j},
\]

while the conditional moments under \( P^* \) are

\[
 E_t^* \left[ \bar{B}_{j}^{H_j}(T) \right] = B_{j}^{H_j}(t)
\]

\[
 Var_t^* \left[ \bar{B}_{j}^{H_j}(T) \right] = \nu_{H_j} (T-t)^{2H_j},
\]

On the other hand, under \( P^* \), the number of jumps \( N(T) \) has the new jump rate \( \lambda^* = \lambda \theta W^{-\gamma} \cdot (\zeta^{-\gamma} + 1) \) and the \( V_k \) have the new density \( f_V^*(x) = (1/\zeta^{-\gamma} + 1) x^{-\gamma} f_V(x) \) (see Kou (2002)).

Let \( S_s \) be one of the \( N \) assets within our market setting and – for sake of tractability – let us assume that this asset is only driven by one of the fractional Brownian motions, \( B_{H_s} \), for some \( s \in j = 1 \ldots J \), i.e. for all \( j \neq s \), the volatility parameters \( \sigma_{sj} \) are zero. For convenience, we simplify notation writing \( S(t), \mu, \sigma, \sigma_W \) and \( H \) instead of \( S_s(t), \mu_s, \sigma_{sk}, \sigma_{W_s} \) and \( H_s \), respectively. Furthermore, for reasons of parsimony, I norm the two-sided exponential distribution so that \( \theta_s \) equals one:

\[
dS(t) = \mu S(t) \, dt + \sigma S(t) \, dB^H(t) + S(t) \, d \left( \sum_{k=1}^{N(T)} (V_k - 1) \right) 
\]
The conditional representation of $S(T)$ under the physical measure $P$ is then:

$$S(T)_{\tilde{\mathbb{F}}t} = S(t) \cdot \exp \left[ \mu (T - t) - \frac{1}{2} \nu_H \sigma^2 (T - t)^{2H} + \sigma (B^H (T) - B^H (t)) \right] \prod_{k=N(t)+1}^{N(T)} V_k$$

The basic pricing equation (16) postulates that under the equilibrium measure $P^*$, today’s value of the stock equals its conditional expected value discounted by the certain market interest rate. Exploiting this and rearranging the terms one receives:

$$e^{r(T-t)} = E^*_t \left[ \exp \left( \mu (T - t) - \frac{1}{2} \nu_H \sigma^2 (T - t)^{2H} + \sigma (B^H (T) - B^H (t)) \right) \cdot \prod_{k=N(t)+1}^{N(T)} V_k \right]$$

$$= E_t \left[ \sigma (B^H (T) - B^H (t)) \right] \cdot E^*_t \left[ \prod_{k=N(t)+1}^{N(T)} V_k \right] \exp \left( \mu (T - t) - \frac{1}{2} \nu_H \sigma^2 (T - t)^{2H} + \sigma (-\gamma \sigma_W \nu_H (T - t)^{2H} + \tilde{\mu}_{T,t}^H) \right)$$

$$= \exp \left( \mu (T - t) - \gamma \sigma_w \sigma_H (T - t)^{2H} + \sigma \tilde{\mu}_{T,t}^H - \lambda (T - t) \theta_W^\gamma \left( \zeta_1^1 - \zeta_1^{1-\gamma} \right) \right)$$

Taking logarithms we receive

$$\mu (T - t) + \sigma \tilde{\mu}_{T,t}^H = r(T - t) + \gamma \sigma_w \sigma_H (T - t)^{2H} + \lambda (T - t) \theta_W^\gamma \left( \zeta_1^1 - \zeta_1^{1-\gamma} \right). \quad (26)$$

On the left hand side, we have the conditional drift of the underlying stock. Here, the classical unconditional drift component is complemented by the drift adjustment resulting from the historic path evolution. In equilibrium, this conditional drift equals the riskfree rate plus two risk premia, one for the diffusion part and one for the jump part of the price process.

It is also possible to rewrite the basic equilibrium condition partially using the parameters under the new measure:

$$\mu (T - t) + \sigma \tilde{\mu}_{T,t}^H = r(T - t) + \gamma \sigma_w \sigma_H (T - t)^{2H} - \lambda^* (T - t) \left( \frac{\zeta_1^{1-\gamma} + 1}{\zeta_1^{1-\gamma} + 1} \right)$$

$$= r(T - t) + \gamma \sigma_w \sigma_H (T - t)^{2H} - \lambda^* (T - t) E^* [V - 1]$$

$$= r(T - t) + \gamma \sigma_w \sigma_H (T - t)^{2H} - \lambda^* (T - t) \zeta_1^1.$$
I conclude this section by providing the conditional representation of \( S(T) \) under the new measure \( P^* \) applying the equilibrium condition above:

\[
S(T)|_{\delta t} = S(t) \cdot \exp \left[ r(T-t) - \frac{1}{2} \nu_H \sigma^2 (T-t)^{2H} - \lambda^*(T-t) \zeta^*_t \right] \cdot \exp \left[ \sigma (\bar{B}^H(T) - \bar{B}^H(t)) \right] \prod_{k=N(t)+1}^{N(T)} \bar{V}_k,
\]

where \( \sigma (\bar{B}^H(T) - \bar{B}^H(t)) \sim_t N \left( 0, \nu_H \sigma^2 (T-t)^{2H} \right) \) and \( \bar{V}_k \) are independent identically distributed so that \( \bar{Y} = \ln \bar{V} \) has an asymmetric double exponential distribution with the density

\[
f^*_Y(y) = p^* \eta^*_1 e^{-\eta^*_1 y} 1_{\{y \geq 0\}} + q^* \eta^*_2 e^{\eta^*_2 y} 1_{\{y \leq 0\}}
\]

with the risk-adjusted parameters

\[
p^* = \frac{p \eta_1}{(\eta_1 + \gamma)(\zeta - \gamma + 1)}, \quad q^* = \frac{q \eta_2}{(\eta_2 - \gamma)(\zeta - \gamma + 1)}, \quad \eta^*_1 = \eta_1 + \gamma \text{ and } \eta^*_2 = \eta_2 - \gamma.
\]

### IV European Option Prices

In this section, I value a European call option \( C_S(t, T) \) that matures at time \( T \), has strike price \( K \) and is written on the stock \( S \).

\[
C_S(t, T) = e^{-r(T-t)} E^*_t \left[ C_S(T, T) \right]. \tag{27}
\]

Adjusting the notation of Kou (2002) to our fractional jump-diffusion case, I define:

\[
\Upsilon(m, s, \lambda, p, \eta_1, \eta_2; a, t, T) = P^* \{ Z(T) \geq a | \mathcal{F}_t \} \tag{28}
\]

where \( Z(T) = Z(t) + X(t, T) + \sum_{i=N(t)+1}^{N(T)} Y_i \), the conditional distribution of \( X(t, T) \) is normal, i.e. \( X(t, T) \sim_t N(m, s) \), \( Y \) has a double exponential distribution with density

\[
f_Y(y) = p \eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q \eta_2 e^{\eta_2 y} 1_{\{y \leq 0\}}\text{ and } N(t) \text{ is a Poisson process with rate } \lambda.
\]

Given the conditional representation of \( S(T) \) under the new measure \( P^* \), one receives the
price of the European call option:

\[ C_S(t,T) = S(t) \Upsilon \left( r(T-t) + \frac{1}{2} \nu_H \sigma^2(T-t)^{2H} - \lambda^*(T-t) \zeta_1^*, \nu_H \sigma^2(T-t)^{2H}, \right) \]

\[ -Ke^{-r(T-t)} \Upsilon \left( r(T-t) - \frac{1}{2} \nu_H \sigma^2(T-t)^{2H} - \lambda^*(T-t) \zeta_1^*, \nu_H \sigma^2(T-t)^{2H}, \right) \]

where

\[ \bar{p}^* = \frac{p^*}{1 + \zeta^*} \cdot \frac{\eta_1^*}{\eta_1^* - 1} \]

\[ \bar{\eta}_1^* = \eta_1^* - 1 \]

\[ \bar{\eta}_2^* = \eta_2^* + 1 \]

\[ \bar{\lambda}^* = \lambda^* \cdot (\zeta^* + 1) \]

\[ \zeta^* = \frac{p^* \eta_1^*}{\eta_1^* - 1} + \frac{q^* \eta_2^*}{\eta_2^* + 1} - 1 \]

By means of the put-call parity, the price of the corresponding put option is given by:

\[ P_S(t,T) = Ke^{-r(T-t)} \left( 1 - \Upsilon \left( r(T-t) - \frac{1}{2} \nu_H \sigma^2(T-t)^{2H} - \lambda^*(T-t) \zeta_1^*, \nu_H \sigma^2(T-t)^{2H}, \right) \right) \]

\[ -S(t) \left( 1 - \Upsilon \left( r(T-t) + \frac{1}{2} \nu_H \sigma^2(T-t)^{2H} - \lambda^*(T-t) \zeta_1^*, \nu_H \sigma^2(T-t)^{2H}, \right) \right) \]

The proof of these formulae is similar to the one given in Kou (2002) and is available upon request from the author.

The formulae can be further investigated considering three special cases.
First, if fractional Brownian motion is simplified to classical Brownian motion, i.e. if $H = \frac{1}{2}$, we receive the formulae of Kou (2002): The variance reduction factor $\nu_H$ as well as the fractional variance exponent $2H$ both equal one.

Second, if one neglects the possibility of jumps and drives $\lambda$ towards zero, the equations turn into fractional Black-Scholes formulae which have already been derived by Rostek (2009), however, only in a setting where investors are risk-neutral and not in a general preference-based equilibrium context:

$$C^\{\lambda=0\}_S(t, T) = S(t)N(d^H_1) - Ke^{-r(T-t)}N(d^H_2),$$  \hspace{1cm} (31)

$$P^\{\lambda=0\}_S = Ke^{-r(T-t)}N(-d^H_2) - S(t)N(-d^H_1).$$  \hspace{1cm} (32)

where

$$d^H_1 = \frac{\ln(S(t)/K) + r(T-t) + \frac{1}{2}\nu_H\sigma^2(T-t)^{2H}}{\sqrt{\nu_H}\sigma(T-t)^{H}},$$  \hspace{1cm} (33)

$$d^H_2 = \frac{\ln(S(t)/K) + r(T-t) - \frac{1}{2}\nu_H\sigma^2(T-t)^{2H}}{\sqrt{\nu_H}\sigma(T-t)^{H}} = d^H_1 - \sqrt{\nu_H}\sigma(T-t)^{H}$$  \hspace{1cm} (34)

and $N(\cdot)$ is the cumulative standard normal distribution function.

The interesting issue about this – in fact less general – formula is that it really helps to understand that the fractional part of the model is a factor that separately explains the so called term structure of volatility. While with respect to moneyness, a fractional "no-jump"-diffusion model, will show flat curves, the maturity effect can be clearly identified and the term structure of volatility has an inverse shape in the case of persistence and a normal upward-sloping shape in the case of anti-persistence. Figure 3 plots the Black-Scholes implied volatilities by model option prices with fictitious parameters and different levels of Hurst parameters.

Third, let us consider the special case of no jumps and independent increments, i.e. $\lambda = 0$
and $H = \frac{1}{2}$. If we specialize the last formula further replacing $\nu_H$ and $H$ by 1 and $\frac{1}{2}$, respectively, we obtain the Black–Scholes pricing formulae for the European call and put option.

Besides mathematical accuracy, the derived formulae can succeed in describing reality in a fair way. To show this, I use the illustration via the implied volatility surface: On the one hand side, I take option prices for different levels of the stock price (i.e. varying moneyness) and different time to maturities available in the market for a large German stock and on the other hand I calculate option prices by the model above fitted to market data. For both data sets, the Black-Scholes pricing formula is inverted to retrograde to implied volatility. Doing so, one obtains the pictures of Figure 4. This figure plots real market data implied volatilities in the upper picture as well as the volatilities implied by the prices of the fitted model in the lower picture.
V Conclusion

I derived closed-form pricing formulae for European options when the market consists of assets driven by geometric fractional Brownian motion and double-exponential jumps. I introduced a single-period trading model à la Brennan (1979) assuming risk-averse investors. The formulae include the results of Kou (2002), Rostek (2009) and Black and Scholes (1973) and Merton (1973) as special cases and thereby provide a coherent theory. By capturing serial correlation and non-normality of returns, the most important time-series properties are modeled. As a consequence, the option pricing model implies a non-flat surface of implied Black-Scholes volatilities. It hence remedies the two major non-conformities of the classical theory with real market data.

A Appendix: The conditional distribution of fractional Brownian motion

The following representation is a minor modification of two basic results due to Gripenberg and Norros (1996) and Nuzman and Poor (2000). It states that for a general Hurst parameter $H$, the conditional distribution of $B_H^T$ based on the observation up to time $t$ is normal with the following moments:

\[
E_t[B_H^T] = B_H^H(t) + \hat{\mu}_T^H, \tag{35}
\]
\[
Var_t[B_H^T] = \nu_H(T-t)^{2H}, \tag{36}
\]
where $\hat{\mu}_{T,t}$ is an integral depending on the whole historic path up to time $t$.\footnote{Explicitly, $\hat{\mu}_{T,t}^H$ equals $(T-t)^{H+\frac{1}{2}} \int_{-\infty}^{t} g(T,t,s) \, ds$, where $g(T,t,s) = \frac{\sin(\pi (H-\frac{1}{2}))(B^H(s) - B^H(t))}{\pi (t-s)^{H+\frac{1}{2}} (T-s)^{\frac{1}{2}(T-s)}}$. For details see Gripenberg and Norros (1996) and Nuzman and Poor (2000).} While the conditional first moment clearly depends on the realized path and has to be calculated by evaluating the past, the conditional second moment only depends on the forecasting horizon $T - t$ and the factor $\nu_H$. Figure 2 in section III depicts the values of $\nu_H$ for varying Hurst parameters.
Figure 4: Implied volatility surface of the fractional double-exponential jump-diffusion model.
References


Shiryayev, A.N., 1998, On arbitrage and replication for fractal models, research report 20, MaPhySto, Department of Mathematical Sciences, University of Aarhus, Denmark.