Optimal Portfolio Allocations with Hedge Funds*

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Abstract

This paper analyzes optimal investment decisions, in the presence of non-redundant hedge funds, for investors with constant relative risk aversion. Factor regression models with option-like risk factors and no-arbitrage principles are used to identify and estimate the market price of hedge fund risk, the volatility coefficients of hedge fund returns and the correlation between hedge fund and market returns. Timing ability causes stochastic fluctuations in these return characteristics. Outside investors optimally hold hedge funds for diversification purposes and are motivated to hedge fluctuations in return components caused by timing ability. The paper examines the portfolio structure and behavior and the impact of timing and selection abilities. Incorporating carefully selected hedge fund classes in asset allocation strategies can be a source of economic gains.

Keywords: Asset allocation, Hedge Funds, Performance Measurement, Market Timing, Market Price of Risk, Economic Value of Hedge Funds.

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1 Introduction

This paper examines the relevance of hedge funds as an additional asset class for outside investors. The paper first proposes a regression-based method, which uses market and hedge fund data at different frequencies, to retrieve the market price of idiosyncratic hedge fund risk. It then derives the optimal asset allocation in the presence of non-redundant hedge funds. The regression method is applied to a universe of hedge fund categories in the HFRI database. A class of hedge funds with significant market timing ability is identified and its impact on the optimal asset allocation examined. Inclusion of this hedge fund category in the allocation policy is shown to be economically beneficial.

This study is partly motivated by pension plans and institutional investors who have sought to improve performance by adding alternative asset classes such as hedge funds, private equity, emerging markets and real estate to their portfolios. Some pension funds have even announced an intent to switch their equity management from a constrained style, with fixed parameters, to an active unconstrained style. Two sets of questions are prompted by this evolution in philosophies. The first one concerns the extent to which the various strategies pursued by hedge funds help pension funds and other investors to increase their portfolio returns without inducing a commensurate increase in risk. The second one concerns the optimal exposure to these strategies. An important aspect in studying these questions is the long-term perspective that should prevail in managing pension funds. Indeed, it is well known that simple analysis based on tactical asset allocation conducted on a period-by-period basis will only provide satisfying answers if return characteristics, such as means and variances, are constants or if investors exhibit myopic behavior. Another salient aspect is the nonlinear structure of hedge fund returns. Empirical studies documenting this behavior include Fung and Hsieh (1997, 2000, 2002), Agarwal and Naik (2004), Chan et al. (2005), Diez and Garcia (2009), Bilio et al. (2006), Chen (2007) and Chen and Liang (2007). Theoretical justification follows from Merton (1980) and Dybvig and Ross (1985), who show that the market timing ability of a hedge fund manager will lead to an additional risk factor with option-like features. The theory, as well as the empirical evidence, warrant a reexamination of optimal asset allocation rules in the presence of hedge fund returns.

Issues regarding the nonlinear structure of asset returns are also examined by Glosten and Jagannathan (1994). Their study recommends regression models with option-like factors to analyze the performance of portfolios that include assets with nonlinear payoffs. They show that managed portfolios of nonlinear contracts can be approximated by a limited number of options written on the market index. In their work, option factors are not introduced in the regression to measure market timing ability, but rather to capture the risk exposure of portfolios with nonlinear contracts.

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1 Models with option payoffs in stochastic discount factors also emerge in other contexts (see Vanden (2004)).
2 Linear stochastic discount factor models with unbounded return distributions are negative on certain events.
Motivation for this approach can be found in the earlier work of Jagannathan and Korajczyk (1986) showing that significant timing regression coefficients do not necessarily reflect superior information or ability, but may be the consequence of portfolio holdings generating option-like payoffs. They show, for instance, that fairly simple strategies like an option buy-write strategy will be detected as market timing ability, if the timing-ability test proposed by Henriksson and Merton (HM) (1981) is performed. In contrast, the same test will misinterpret a hedging strategy such as “writing a covered call” as inferior timing ability and superior selection ability. The goal of the present study is not to provide a cleaner separation between market timing/selection ability and derivatives trading, but to examine the effects of option-like risk factors on the optimal asset allocation. In that regard, even though the motivations for the models of Glosten and Jagannathan (1994) and Henriksson and Merton (1981) are different, for our purposes the two models are observationally similar. In here, these regression models are used to the extent that they permit the identification of the market prices of risk factors idiosyncratic to hedge fund returns. As hedge funds are treated as a non-redundant asset class, these risks are priced in equilibrium and affect the portfolio allocation of an outside investor.

If hedge fund returns are generated by managed portfolios of traded assets using public information, they are redundant. This remains true even if derivatives are used or the investment policy of the hedge fund manager is stochastic and time-varying and therefore the volatility of hedge fund returns is nonlinear. Managed funds of this type are of no value to outside investors, as return patterns could be duplicated on own account. Non-redundant hedge funds, in contrast, expand the investment opportunity set and the space of attainable return distributions. Economic arguments for non-redundancy include the ability of certain hedge funds to access investment opportunities that are outside the class of traded financial securities. From these simple considerations, it is clear that only non-redundant hedge funds are of interest as an additional asset class. But in order to determine the optimal asset allocation policy in the presence of these non-redundant managed portfolios, the market price of unspanned hedge fund risk has to be identified. This is not an easy task, because hedge fund returns are reported on a relatively low, monthly frequency. Optimal asset allocation, in contrast, requires decisions at a much higher frequency and is influenced by the behavior of the market price of hedge fund risk within the observation window.

Our main contributions are as follows. First, in order to derive the structure of hedge fund strategies and returns, the previous regression models with option factors are embedded in a con- and cannot be used to price portfolios of nonlinear contracts. Setting the stochastic discount factor equal to the positive part of a linear stochastic discount factor is econometrically challenging, but provides statistically significant improvements when the performance of hedge funds is studied (see Bailey et al. (2004)). Option payoffs, which form a basis for various spaces of contingent claims, can be used to approximate nonlinear stochastic discount factors.

Even though terminology is ambiguous, for ease of exposition, various components of excess returns will be said to reflect timing or selection ability.
tinuous time setup. This extended regression setup can be seen as the continuous time limit of the generalized Henriksson-Merton model of Goetzmann, Ingersoll and Ivkovic (2000), who consider the effects of timing windows that are shorter than the observation frequency. Our continuous time setting allows for timing windows of any length. Furthermore, our regression analysis provides valuable information at higher frequencies, even if hedge fund returns are observed only at a monthly frequency.

Second, for any given hedge fund strategy, the regression model implies the hedge fund return volatility, its correlation with market returns and the market price of hedge fund risk. Timing ability generates a contingent claim with nonlinear payoff written on the market portfolio (the timing option). The hedge fund return volatility, and the correlation with the market, are shown to be related to the Delta of the timing option value. The market price of hedge fund risk has two parts. The first part measures the selection ability of the hedge fund. The second part is related to its timing ability. It depends on the Theta and the Theta of Delta of the timing option. Thus, hedge fund return characteristics, such as volatility, correlation coefficient and market price of risk, all depend on realized market returns. They evolve stochastically over the timing window and exhibit nonlinear behavior relative to market returns. They also exhibit non-Markovian behavior: hedge fund timing-strategies naturally induce path-dependence in returns.

Third, optimal asset allocation policies include investments in hedge funds. The intrinsic reason why hedge funds are valued is because of the diversification services they provide (hedge funds are a non-redundant asset class). Hedge fund timing strategies influence the allocation to equities. With Henriksson-Merton type of strategies, the demand for equities has a mean-variance as well as a dynamic hedging component, even when market returns themselves are i.i.d.. The hedging component, in the i.i.d. market return case, is entirely motivated by the timing behavior of hedge fund managers which induces stochastic fluctuations in the implied market price of risk. A detailed analysis of the structure of this component and of the asset allocation is carried out.

Fourth, empirical estimation of hedge fund returns for the HFRI database identifies Macro Systematic Diversified as a category of hedge funds with significant timing return components. Asset allocation policies based on this category are studied. Optimal investments in hedge funds and in the market vary over the timing window, because of variations in the market price of hedge fund risk. The patterns of investments in the two asset classes are examined under various market scenarios (bull and bear markets). The economic value of Macro Systematic Diversified is also assessed. Investments in this asset class are found to improve certainty-equivalents by significant amounts. Gains of the order of 17.25% are recorded for an investor with risk aversion equal to 4 and investment horizon of 3 years.

Lastly, various extensions of this base model are provided. As hedge funds can not be held

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short, a no short sales constraint is introduced and its effects on the investment policy are studied. The impact of the alternative risk factors proposed by Hasan Hodzic and Lo (2005) is also examined. The structure of the optimal portfolio is further studied when Fama-French factors (value, size and momentum) are added. These factors are treated either as investable or non-investable portfolios and the corresponding portfolio weights are analyzed.

As a final note, it is also relevant to mention that market prices of hedge fund risk determined in this paper are not only important for asset allocation, but also provide a new class of performance measures. In fact, if hedge funds are non-redundant assets, the market price of hedge fund risk implied by selection and timing ability, generalizes the notion of total performance measure introduced in Treynor and Mazuy (1966). In a general setting with nonlinear timing regression functions, this generalized total performance measure is path-dependent and stochastic. However, for i.i.d. returns and quadratic timing factor as in Treynor and Mazuy, the market price of idiosyncratic hedge fund risk becomes constant and corresponds to their total performance measure.

Optimal portfolio selection in the presence of active funds with both selection and timing ability has not been considered in the previous literature. The paper that comes closest is Cvitanic, Lazrak, Martellini and Zapatero (2006). They focus on asset allocation in the presence of a potentially mis-priced stock. Investors are assumed to have incomplete information with Gaussian priors about the expected market return and the information ratio of the stock. Market information is then used to update the initial priors. As the perceived investment opportunity set is stochastic, learning about parameter uncertainty introduces a horizon-dependent dynamic hedging motive. The perceived mispricing premium can be reinterpreted as a time-varying premium due to selection ability. The present paper allows for selection and timing ability. It shows that timing introduces other components in the market price of risk and the return volatility coefficients, determined by the Greeks of the timing option. The resulting structure is non-Markovian, but can be estimated using market returns at a higher frequency, relying on a classic regression approach. The timing-induced hedging demand depends on the timing window and not on the investment horizon. Its features differ from those found in hedging demands associated with selection ability.

The paper is organized as follows. Section 2 introduces the basic model and shows how to derive the market price of risk idiosyncratic to hedge funds. Section 3 specializes to i.i.d. returns. The optimal portfolio is derived and analyzed in Section 4, for the simple setting of Section 3. First, the unconstrained and constrained optimal portfolios are derived. Then, the case of additional risk factors is examined. The estimation of the model parameters is performed in Section 5. Results are presented and discussed in Section 6. Conclusions appear in Section 7. Appendix A describes hedge fund return characteristics in settings with stochastic opportunity sets. Appendix B derives the market price of hedge fund risk for Henriksson-Merton strategies. Aspects pertaining to the empirical implementation of the models are presented in Appendix C. Proofs are in Appendix D.
2 Hedge Fund Regression Models and Market Prices of Risk

This section develops a general methodology for identifying hedge fund return characteristics from data. Section 2.1 reviews the regression models studied in the literature. Section 2.2 explains how to embed a continuous time hedge fund return process in the regression framework and how to identify relevant return characteristics.

2.1 Market Timing Regressions and Tests

Market timing ability has been mainly tested under the assumptions of two particular models, the Treynor-Mazuy (TM) model (Treynor and Mazuy (1966)) and the Henriksson-Merton (HM) model (Henriksson and Merton (1981)). The two tests assume that normal returns are described by a market model. They differ by a term added to describe abnormal returns. The TM model includes a quadratic term in (excess) market returns

\[ r_{pt+1} = \alpha_p + \beta_p r_{mt+1} + \gamma (r_{mt+1})^2 + \nu_{t+1} \]

where \( r_{pt+1} \) and \( r_{mt+1} \) are excess returns over a risk-free rate. The HM model adds an option component in the market regression model

\[ r_{pt+1} = \alpha_p + \beta_p r_{mt+1} + \gamma (r_{mt+1})^{+} + \nu_{t+1} \]

where \( (r_{mt+1})^{+} \equiv \max \{0, r_{mt+1}\} \).

The absence of market timing ability is tested by the null that \( \gamma = 0 \). These basic regressions have been extended in various directions. The first obvious extension is to change the model for normal returns, the most frequently used being the Fama-French three-factor model. The next natural extension is to use conditional factor models. For instance, Ferson and Schadt (1996) conduct the TM and HM analysis in conditional settings, by allowing portfolio managers to use public information variables. In these frameworks, components capturing the interaction between market and conditioning variables appear in the regressions along with the market and market timing variables.

Two other extensions are relevant for setting up our approach. Instead of using a quadratic or a maximum function, one can include a general nonlinear function \( f \) of the market returns

\[ r_{t+1}^{p} = \alpha_p + \beta_p r_{t+1}^{m} + \gamma f (r_{t+1}^{m}) + \nu_{t+1}^{p}. \]

When this nonlinear exposure to the market is captured by a limited number of options, the regression becomes

\[ r_{t+1}^{p} = \alpha_p + \beta_p r_{t+1}^{m} + \sum_{i=1}^{K} \gamma_i \max \{r_{t+1}^{m} - \kappa_i, 0\} + \nu_{t+1}^{p}. \]
One restriction of the previous models is the assumption that the trading and the return measurement frequencies coincide. If returns are measured monthly and the portfolio manager trades several times during the month, the previous regressions are misspecified. Often, and this is the case for hedge funds, it is impossible to gather high-frequency data for managed portfolios. For this reason, Goetzman, Ingersoll and Ivkovic (2000) propose to include an instrument that is correlated with option-like terms in the regression. If portfolio returns are observed at a monthly frequency a natural instrument is the cumulative value of the timing options over the month. If $R_{m}^{m}$ and $R_{f}^{f}$ denote the daily market and risk free returns, then a market timer will realize daily gross returns equal to \( \max\{1 + R_{m}^{m}, 1 + R_{f}^{f}\} \). The cumulative value of these daily timing options can be written as

\[
P_{m}^{m,t+1} = \prod_{\tau \in \text{month}(t,t+1)} \max\{1 + R_{m}^{m,\tau}, 1 + R_{f}^{f,\tau}\} - (1 + r_{m}^{m,t+1}).
\]

The regression becomes

\[
r_{p}^{t+1} = \alpha_{p} + \beta_{p}r_{t+1}^{m} + \gamma P_{t+1}^{m} + \nu_{t+1}^{p}.
\]

In other words, they replace the value of a monthly timing option on the market by a rolling account through the month of the gains generated by daily market timing options. Monthly observations of portfolio returns and daily observations of market and riskfree returns are needed to run this regressions and test timing ability.

The next section develops a continuous-time model where timing can occur at any instant and where the function $f$ of cumulated returns over the return measurement window is left general. Such a setting has several advantages. Recognizing that timing can occur at any instant is closer to reality. Moreover, the availability of high-frequency data makes it possible to compute cumulated market returns over intervals smaller than a day. Lastly, the continuous-time approach will eventually enable us to reach the goal stated at the outset of the paper, namely the determination of the optimal asset allocation rule in the presence of investable hedge funds.

2.2 Market Timing, Nonlinear Factors and Return model

This section embeds the timing regression in a continuous time framework. As in discrete time, the timing regression seeks to identify the market value of the timing skill, i.e., the value of the timing option. At the same time, it provides a filter that permits the identification of a hedge fund return process consistent with discrete information about fund performance and continuous information about market returns. This identification proceeds in two steps. The first step prices the timing option and finds its replicating portfolio. The second step substitutes the replicating policy in the timing regression to derive the implied parameters of the return process.

The hedge fund model is as follows. Consider a hedge fund manager who can invest in two
types of assets, publicly traded assets (the market portfolio) and private assets. Let \((\pi_h,e, \pi_h,p)\) be the manager’s trading strategy in the two risky asset classes and let \(t, \tau)\) be the trading window. The cumulative excess return \(R_{t,\tau}^h\) of the hedge fund over this trading window is

\[
R_{t,\tau}^h = \int_t^\tau \pi_{h,e}^v dR_v^e + \int_t^\tau \pi_{h,p}^v dR_v^p
\]  

(6)  

where

\[
dR_v^e = \sigma_v^e (\theta_v^e dv + dW_v^e) \quad \text{and} \quad dR_v^p = \sigma_v^p (\theta_v^p dv + dW_v^p)
\]  

(7)  

represent the instantaneous excess return of the market portfolio \((dR_v^e)\) and the instantaneous excess return of private assets \((dR_v^p)\). The parameters \(\sigma_v^e, \sigma_v^p\) are the volatilities of market and private asset returns and \(\theta_v^e, \theta_v^p\) are the prices of risk associated with the two asset classes (the risk premia per unit risk). The components \(W_v^e, W_v^p\) are independent Brownian motions representing economic shocks to the market \((W_v^e)\) and to the private assets \((W_v^p)\). Excess returns are measured relative to the riskfree rate \(r\). It is assumed that market coefficients \((r, \theta_v^e, \sigma_v^e)\) just depend on market risk \(W_v^e\), whereas \((\theta_v^p, \sigma_v^p)\) depend on all risks \((W_v^e, W_v^p)\). In this setup the hedge fund is a non-redundant asset as it provides access to nontraded risk \(W_v^p\).

The portfolio strategies of hedge fund managers are unobservable to the public. Likewise, the return on private assets is unobservable to outsiders. Let \(\mathcal{F}_\tau^e\) be the information conveyed by the market at time \(\tau\) (the trajectory of \(W_v^e\) up to time \(\tau\)). The conditional expected hedge fund return based on realized market information is

\[
E_t \left[ R_{t,\tau}^h \bigg| \mathcal{F}_\tau^e \right] = E_t \left[ \int_t^\tau \pi_{h,e}^v dR_v^e \bigg| \mathcal{F}_\tau^e \right] + \int_t^\tau \alpha_t,v dv
\]

(8)

where \(\alpha_t,v = E_t[\pi_{h,p}^v \sigma_v^p | \mathcal{F}_\tau^e]\).  

If the manager pursues a known pure buy-and-hold strategy \(\pi_{h,e}^v = \beta_t\) for \(v \in [t, \tau)\) where \(\beta_t\) depends on public information \(\mathcal{F}_\tau^e\), then

\[
E_t \left[ \int_t^\tau \pi_{h,e}^v dR_v^e \bigg| \mathcal{F}_\tau^e \right] = \beta_t R_{t,\tau}^e.
\]

If she pursues a pure timing strategy \(\pi_{h,e}^v = \gamma h (\widehat{R}_{t,\tau})\), where \(\widehat{R}_{t,\tau}\) is a forecast of excess market return over the timing window that is correlated with \(R_{t,\tau}^e\), then

\[
E_t \left[ \int_t^\tau \pi_{h,e}^v dR_v^e \bigg| \mathcal{F}_\tau^e \right] = \gamma_t f \left( t, R_{t,\tau}^e \right).
\]

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5Private assets are those that are not accessible to the general public. A typical example is private equity. Dynamic portfolio strategies requiring specialized knowledge, execution ability or information, constitute another example.

6It is also assumed that \(\sigma_{h,p}^v \equiv \pi_{h,p}^v \sigma_v^p\) is \(\mathcal{F}_\tau^e\)-measurable. As discussed later, this assumption is weaker than those found in standard timing models.
for some function $f(\cdot)$ and timing coefficient $\gamma_t$. Hence, under the null hypothesis of mixed market timing strategies and buy-and-hold strategies, the following regression

$$R_{t,\tau}^h = \mathbf{E}_t \left[ R_{t,\tau}^h \bigg| \mathcal{F}_\tau^e \right] + \left( R_{t,\tau}^h - \mathbf{E}_t \left[ R_{t,\tau}^h \bigg| \mathcal{F}_\tau^e \right] \right) = \int_t^\tau \alpha_{t,v} \, dv + \beta_t \int_t^\tau dR_v^e + \gamma_t f(t, R_{t,\tau}^e) + \int_t^\tau \sigma_v^{h,p} \, d\tilde{W}_v^p \quad (9)$$

where $d\tilde{W}_v^p \equiv dW_v^p + \left( \theta_v^e - \alpha_{t,v}/\pi_v^{h,p} \sigma_v^p \right) \, dv$ is a Brownian motion with respect to market information, characterizes hedge fund returns.\(^7\) This is a nonlinear factor regression model with error $\epsilon_{t,\tau} = \int_t^\tau \sigma_v^{h,p} \, d\tilde{W}_v^p$. The volatility coefficient in $\epsilon_{t,\tau}$ is $\sigma_v^{h,p} = \pi_v^{h,p} \sigma_v^p$.

Hedge fund outsiders do not observe the instantaneous returns on the private investments undertaken by the hedge fund, nor do they observe the specific asset allocation strategies implemented. They nevertheless receive reports of hedge fund returns at regular time intervals. If $\{t_n : n = 1, \ldots, N\}$ is the set of reporting dates, they observe the sequence of returns $\left\{ R_{t_{n-1},t_n}^h : n = 1, \ldots, N \right\}$. This information, along with market information, can be used in order to estimate the parameters and functions $(\alpha, \beta, \gamma, f(t, \cdot), \sigma_v^{h,p})$ in the return process (9). Knowledge of this structure, in turn, can be exploited to retrieve some of the more fundamental characteristics of the hedge fund return process, parameters that are of interest to an investor with the ability to revise his/her portfolio allocation within the reporting window. To convert the regression to a model that can be used by a publicly-informed investor operating at a high frequency, proceed as follows.

First, let $C_s(t, \tau)$ be the value in the money market numeraire, at time $s \in [t, \tau)$, of a claim with terminal payoff $f(t, \tau, R_{t,\tau}^e)$ (the timing claim). Standard valuation principles can be invoked to conclude that

$$C_s(t, \tau) = \mathbf{E}_s \left[ Z_{s,\tau}^e f \left( t, \tau, \int_t^\tau dR_v^e \right) \right] \quad (10)$$

where

$$Z_{s,\tau}^e \equiv \exp \left( -\int_s^\tau \theta_v^e dW_v^e - \frac{1}{2} \int_s^\tau (\theta_v^e)^2 \, dv \right) \quad (11)$$

is the state price density (stochastic discount factor) for traded risk $W^e$, using the money market account as the numeraire.\(^8\) Moreover, (see the proof of Proposition 1 in Appendix D for details)

$$f(t, \tau, R_{t,\tau}^e) = C_t(t, \tau) + \int_t^\tau \Delta_v(t, \tau) \, dR_v^e \quad (12)$$

\(^7\) Under the assumption that the volatility $\sigma_v^{h,p}$ in (9) is $\mathcal{F}_\tau^e$-measurable, the process $\tilde{W}_v^p$ is a Brownian motion with respect to $\mathcal{F}_\tau^e$. To see this, note that $\alpha_{t,v} = \pi_v^{h,p} \sigma_v^p \mathbf{E}_t \left[ \theta_v^e \big| \mathcal{F}_\tau^e \right]$ and $dW_v^p = dW_v^e + \left( \theta_v^e - \mathbf{E}_t \left[ \theta_v^e \big| \mathcal{F}_\tau^e \right] \right) \, dv$. It is then clear that $\tilde{W}_v^p$ is a continuous martingale with respect to $\mathcal{F}_\tau^e$ with initial value $\tilde{W}_v^p \big|_0 = 0$ and quadratic variation $\left[ \tilde{W}_v^p \big|_0 \right] = [W_v^e]_\tau = v$. Levy’s Theorem can be invoked to conclude that $\tilde{W}_v^p$ is an $\mathcal{F}_\tau^e$-Brownian motion (see Karatzas and Shreve (1988)).

\(^8\) The hedge fund model is formulated in terms of excess returns. Asset values are therefore automatically expressed in the money market numeraire. The state price density in this numeraire is $Z_{s,\tau}^e$. In the special case of constant interest rate the money market numeraire is the same as a bond numeraire.
where
\[ \Delta_v(t, \tau) \equiv \frac{\mathcal{D}_v^e C_v(t, \tau)}{\sigma_v^e} \tag{13} \]
is the Delta of the claim, i.e., its sensitivity to the underlying market uncertainty, and where \( \mathcal{D}_v^e \) stands for the Malliavin derivative with respect to \( W^e \).\(^9\) The portfolio replicating the timing claim is \( \Delta_v(t, \tau) \).

Second, use Ito’s lemma applied to the regression and replicating portfolio, to deduce that
\[ dR_{t,v}^h = \alpha_{t,v} dv + \beta_t dR_v^e + \gamma_t \left( \Theta_s(t, v) + \int_t^v \theta_s(t, v) dR_v^e \right) dv + \Delta_s(t, v) dR_v^e + \sigma_{v \Delta}^e \tilde{d}W_p^e \tag{14} \]
where
\[ \Theta_s(t, v) \equiv \frac{\partial C_s(t, v)}{\partial v} \quad \text{and} \quad \theta_s(t, v) = \frac{\partial \Delta_s(t, v)}{\partial v} \tag{15} \]
are the Theta of the claim (\( \Theta_s(t, v) \)) and the Theta of the Delta (\( \theta_s(t, v) \)), i.e., their respective sensitivities with respect to the horizon of the timing window. Hence, relative to market information, excess hedge fund returns satisfy
\[ dR_{t,v}^h = \sigma_{t,v}^{h,e}(\theta_v^e dv + dW_v^e) + \sigma_{t,v}^{h,p}(\theta_v^h dt + d\tilde{W}_v^p) \tag{16} \]
where \( \sigma_{t,v}^{h,e} = \pi_{t,v}^{h,e} \sigma_v^e \), \( \sigma_{t,v}^{h,p} = \pi_{t,v}^{h,p} \sigma_p^p \) and \( \theta_v^h \) is the market price of hedge fund risk. Expressions for \( \sigma_{t,v}^{h,e} \) and \( \theta_v^h \), which follow from (14), are presented in the next proposition.

**Proposition 1.** The market volatility coefficient of the hedge fund return is
\[ \sigma_{t,v}^{h,e} = \kappa_{t,v} \sigma_v^e \quad \text{where} \quad \kappa_{t,v} \equiv \beta_t + \gamma_t \Delta_v(t, v). \tag{17} \]
The market price of idiosyncratic hedge fund risk (total performance measure) and the correlation coefficient between market and hedge fund returns are
\[ \theta_v^h = \theta_v^{h,S} + \theta_v^{h,T} \equiv \frac{\alpha_{t,v}}{\sigma_{t,v}^{h,p}} + \gamma_t \left( \Theta_s(t, v) + \int_t^v \theta_s(t, v) dR_v^e \right) / \sigma_{t,v}^{h,p} \tag{18} \]
\[ \theta_v = \frac{d[R_v^e, R_v^h]}{\sqrt{d[R_v^e] \sqrt{d[R_v^h]}}} = \frac{\kappa_{t,v}}{\sqrt{\kappa_{t,v}^2 + \left( \sigma_{t,v}^{h,p} / \sigma_v^e \right)^2}} \tag{19} \]
The arbitrage-free conditional state price density in the market with non-redundant, actively managed fund, over the timing window \([t, \tau)\), is
\[ \xi_{t,v} = Z_{t,v}^e \exp \left( - \int_t^\tau r_s ds - \int_t^v \theta_v^h d\tilde{W}_s^p - \frac{1}{2} \int_t^v \left( \theta_v^h \right)^2 ds \right) \tag{20} \]
for \( v \in [t, \tau) \), where \( Z_{t,v}^e \), defined in (11), is the stochastic discount factor for traded risk in the money market numeraire. This state price density is the unique one consistent with the regression estimates.

The price of hedge fund risk $\theta_{t,v}^h$ can be interpreted as a generalization of the total performance measure studied by Treynor and Mazuy (1966). It has the two components $\theta_{t,v}^h = \theta_{t,v}^{h,S} + \theta_{t,v}^{h,T}$ where

$$\theta_{t,v}^{h,S} \equiv \frac{\alpha_{t,v}}{\sigma_{h,p}^v}$$

and

$$\theta_{t,v}^{h,T} \equiv \gamma_t \frac{\Theta_t(t,v) + \int_t^v \vartheta_s(t,v) dR_s^e}{\sigma_{h,p}^v}.$$

The first term, $\theta_{t,v}^{h,S}$, is the Treynor and Black (1973) appraisal (or information) ratio which measures selection ability. The second term, $\theta_{t,v}^{h,T}$, is a measure of timing ability. The distinction between selection and timing ability relies on Admati, Bhatacharya, Pfleiderer and Ross (1986) and Grinblatt and Titman (1989b). As discussed by these authors, a hedge fund manager has selectivity information if he/she has information that is uncorrelated with market information. This knowledge creates value reflected in the alpha $\alpha_{t,v} \neq 0$. Timing information, in contrast, correlates the unobserved portfolio policy of the fund manager with future returns. Special examples of timing measures were introduced by Treynor and Mazuy (1966), Merton (1981) and Henriksson and Merton (1981). These classic timing measures, unlike the generalization above, assume that the timing window corresponds to the inter-observation window.\(^{10}\)

The reward for idiosyncratic hedge fund risk vanishes ($\theta_{t,v}^h = 0$) if and only if the contributions of timing and selection offset each other ($\theta_{t,v}^{h,S} = -\theta_{t,v}^{h,T}$). This could occur for a variety of reasons. It arises in particular, when there is neither timing, nor selection ability (i.e., $\alpha_{t,v} = \gamma_t = 0$). In that case, there is no incentive to invest in the hedge fund, which effectively becomes a redundant asset class (details are in Proposition 3). In cases where a reward exists, the market price of risk associated with timing ability $\theta_{t,v}^{h,T}$ depends non-monotonically on the horizon of the timing window. This follows as its Theta, i.e., the sensitivity of the timing factor with respect to the horizon of the timing window, is in general not monotone in horizon. Similarly, Theta will in general be state-dependent. In this case, the Delta of the Theta, $\vartheta_s(t,v)$, does not vanish either. The total performance measure is then time-varying and non-Markovian. It depends on the whole path of excess returns over the timing window. The volatility and correlation coefficients ($\sigma_{t,v}^{h,e}, \varrho_{t,v}$) have similar properties. These complex characteristics of hedge fund returns imply that optimal investments in hedge funds timing the market are necessarily time-varying and depend in a non-trivial manner on the realized return trajectories.

In the particular case of a multiplicative timing claim, Proposition 1 becomes

Corollary 1. Suppose that market timing generates the multiplicative claim

$$f \left( t, \tau, R_{t,\tau}^e \right) \equiv g(\tau-t) F \left( t, R_{t,\tau}^e \right)$$

where $g(\tau-t)$ depends on the timing window alone and $F \left( t, R_{t,\tau}^e \right)$ captures the market return dependence. The market volatility coefficient of the hedge fund return and the market price of

\(^{10}\)See Goetzmann, Ingersoll and Ivkovic (2000) for an analysis of asynchronous timing and observation frequencies.
idiosyncratic hedge fund risk are given by (17) and (18) with

\[ \Delta_v(t, v) = g(v - t) \Delta^F_v(t, v) \]  

(21)

\[ \Theta_t(t, v) = g(v - t) \Theta^F_t(t, v) + \frac{\partial g(v - t)}{\partial v} C^F_t(t, v) \]  

(22)

\[ \vartheta_s(t, v) = g(v - t) \vartheta^F_s(t, v) + \frac{\partial g(v - t)}{\partial v} \Delta^F_s(t, v) \]  

(23)

In these formulas

\[ C^F_s(t, \tau) = E_s \left[ Z_{s,\tau} F \left(t, R^e_{t,\tau}\right) \right] \]  

(24)

is the value at \( s \) of the payoff \( F \left(t, R^e_{t,\tau}\right) \) and \( \Delta^F_v(t, v), \Theta^F_t(t, v), \vartheta^F_s(t, v) \) are the associated Greeks.

Strategies leading to the multiplicative payoff are those involving a scaling factor \( g(\tau - t) \) depending on the timing window. As will be seen shortly, an extended version of Henriksson-Merton model falls in this category.

3 Benchmark Timing Models with Independent Increments

Standard timing models assume i.i.d. market returns, i.e., they postulate that the market portfolio follows a geometric Brownian motion. The market coefficients \( (r, \theta^e, \sigma^e) \) are then constant. They also assume that the other coefficients \( (\alpha_{t,v}, \sigma^{h,p}_{t,v}) \) in the conditional factor regressions are constant over the timing window. In these settings the implied market price of hedge fund risk becomes

\[ \theta^h_{t,v} = \frac{\alpha}{\sigma^{h,p}} + \gamma \left( \Theta_t(t, v) + \int_t^v \vartheta_s(t, v) dR^e_s \right) \sigma^{h,p} \]

where Theta \( \Theta_t(t, v) \) and Theta of Delta \( \vartheta_s(t, v) \) depend on the particular model selected to describe the timing ability of the hedge fund manager.

3.1 Treynor-Mazuy Timing Strategies

The earliest timing model is due to Treynor and Mazuy (TM) (1966). They assume that timing generates the quadratic claim

\[ f \left(t, R^e_{t,\tau}\right) = \left(R^e_{t,\tau}\right)^2. \]  

(25)

This is consistent with a linear timing strategy of the form \( \pi^{h,e}_v = \beta + \gamma R^e_{t,\tau} + w \), where \( w \) is an independent noise term (see Admati, Bhattacharya, Pfleiderer and Ross (1986) and Ferson and Schadt (1996)).

In this context, the implied parameters described in Proposition 1 are in explicit form.
Corollary 2. Consider the Treynor-Mazuy market timing model, with constant market coefficients $(r, \theta^e, \sigma^e)$ and constant parameters $(\alpha, \sigma^{h,p})$. The value of the timing option is

$$C_{s}^{TM}(t, v) = (R_{t,s}^{e})^2 + (\sigma^e)^2(v - s). \quad (26)$$

The hedge fund market volatility coefficient and the correlation with the market portfolio are

$$\sigma_{h,e}^{T,M} = \kappa_{T,M}^{T,M} \sigma^e$$

where

$$\kappa_{T,M}^{T,M} = \beta^{TM} + 2\gamma^{TM} R_{t,v}^{e} \quad (27)$$

$$\theta_{h}^{TM} = \frac{\kappa_{T,M}^{T,M}}{\sqrt{(\kappa_{T,M}^{T,M})^2 + (\sigma^{h,p}/\sigma^e)^2}} \quad (28)$$

The Greeks associated with the quadratic TM timing claim are

$$\Theta_{s}^{TM}(t, v) = (\sigma^e)^2, \quad \vartheta_{s}^{TM}(t, v) = 0. \quad (29)$$

The market price of hedge fund risk (total performance measure) is,

$$\theta_{h}^{TM} = \theta_{h,S,T,M}^{T,M} + \theta_{h,T,T,M}^{T,M} = \alpha/\sigma^{h,p} + \gamma^{TM}(\sigma^e)^2/\sigma^{h,p}. \quad (30)$$

The market volatility coefficient of the hedge fund return, $\sigma_{h,e}^{T,M}$, is proportional to the market volatility with proportionality factor $\kappa_{T,M}^{T,M}$. The proportionality factor is positively related to systematic risk $\beta$ and also depends on the timing ability $\gamma^{TM}$. For long market timers ($\gamma^{TM} > 0$), the hedge fund market volatility coefficient increases (decreases), and therefore the volatility of the hedge fund return increases (decreases), if realized cumulative returns are positive (negative). The reverse holds for short timers ($\gamma^{TM} < 0$). The hedge fund market volatility coefficient is null at times at which the intrinsic exposure of the hedge fund to market risk is offset by the impact of timing ability (i.e., times when $\beta^{TM} + 2\gamma^{TM} R_{t,v}^{e} = 0$). It is persistently null if the hedge fund is intrinsically market neutral ($\beta = 0$) and has no timing ability ($\gamma^{TM} = 0$).

In the presence of market timing the correlation between hedge fund and market returns becomes stochastic. The factors underlying the volatility of the hedge fund return are also determinants of the correlation coefficient. When $\kappa_{T,M}^{T,M}$ is positive (negative), the correlation is positive (negative). An increase in $|\kappa_{T,M}^{T,M}|$ leads to an increase in the absolute value of the correlation coefficient. Moreover, times at which intrinsic market exposure and timing payoff offset each other are times at which the hedge fund return is uncorrelated with market returns. It is also interesting to note that hedge funds that have an intrinsically positive exposure to the market ($\beta > 0$) can become negatively correlated with the market returns in bear markets (i.e., at times when $\kappa_{T,M}^{T,M} < 0$).

Market timing ability increases the market price of hedge fund risk by $\theta_{h,T,T,M}^{h,T,T,M} = \gamma^{TM} \times (\sigma^e)^2/\sigma^{h,p}$, which is positive (negative) if $\gamma^{TM}/\sigma^{h,p}$ is positive (negative). The numerator of the market price of hedge fund risk, $\alpha + \gamma^{TM}(\sigma^e)^2$, corresponds to the classic TM total performance
measure. In the TM model total performance is constant because the Theta of the timing option is the market variance \(\Theta_{TM}^t(t,v) = (\sigma^e)^2\), which is constant by assumption, and because the Delta of Theta is null \(\delta^T_{TM}(t,v) = 0\).

### 3.2 Henriksson-Merton Timing Strategies

Merton (1981) and Henriksson and Merton (HM) (1981) focus on directional trading ability. The trading strategies are of the form \(\pi^h_{v} = \gamma^c\mathbf{1}_{(R^e_{t,\tau} > 0)}\) and the associated timing option pays \(\gamma^p (R^e_{t,\tau})^+\), which corresponds to a call factor. A straightforward extension of HM is to add a directional timing trade corresponding to a put option factor. The resulting general structure that identifies timing ability takes the multiplicative form

\[
f(t, \tau, R^e_{t,\tau}) \equiv g(\tau - t) F\left(t, R^e_{t,\tau}\right) \equiv \sqrt{\tau - t} \left(\gamma^c \left(R^e_{t,\tau} - k^c(\tau - t)\right)^+ + \gamma^p \left(k^p(\tau - t) - R^e_{t,\tau}\right)^+\right)
\]

The coefficient \(\gamma^c\) (resp. \(\gamma^p\)) captures the ability to time bull (resp. bear) markets. A successful timer playing both bull and bear markets \((\gamma^c, \gamma^p > 0)\) effectively holds a straddle (if \(k^c = k^p\)) or a strangle (if \(k^c > k^p\)). The scaling factor \(g(\tau - t) \equiv \sqrt{\tau - t}\) in front of the option payoffs ensures that the implied reward for hedge fund risk remains finite (see Corollary 4). This property is important when it comes to optimal investments. It effectively excludes arbitrage opportunities for outside investors.\(^{11}\) The scaling factor also ensures that the regression model is balanced.\(^{12}\)

In the extended HM model, explicit formulas can be also be derived for the various components of the market price of hedge fund risk.

**Corollary 3.** Consider the extended Henriksson-Merton market timing model with call and put option factors, constant market coefficients \((r, \theta^e, \sigma^e)\) and constant parameters \((\alpha, \sigma^{h,p})\). Let \(\Phi(x)\) be the cumulative normal distribution with density \(\phi(x)\) and define the functions

\[
d(k, v - t) \equiv \frac{k\sqrt{v - t}}{\sigma^e}, \quad e(R; k, t, v, s) \equiv \frac{k(v - t) - R}{\sigma^e\sqrt{v - s}} \quad (31)
\]

\[
l(R; k, t, v, s) \equiv \frac{k(v - t) \left(1 - 2\frac{v - t}{v - s}\right) - R}{2\sigma^e(v - s)^{3/2}} \quad (32)
\]

\(^{11}\)Without scaling, the market price of idiosyncratic hedge fund risk explodes in finite time in certain states. In these events, Arrow-Debreu prices become null and arbitrage opportunities emerge. The investment problem is then ill-posed: the optimal policy is to invest unlimited amounts in the hedge fund.

\(^{12}\)If the regression is unbalanced, the estimators for \(\alpha, \beta\) converge faster than those for \(\gamma^p, \gamma^c\). In this situation, called super-consistency in the literature, timing will be asymptotically negligible and standard statistical inference methods do not apply (see the unit root literature).
The timing call and put option values are

$$C_t(t,v,k_c) = \frac{\sigma^e \phi(d(k_c,v-t)) - k_c \sqrt{v-t} \Phi(-(d(k_c,v-t))}{\sqrt{v-t}}$$ \hspace{1cm} (33)

$$P_t(t,v,k_p) = \frac{k_p \sqrt{v-t} \Phi(d(k_p,v-t)) + \sigma^e \phi(d(k_p,v-t))}{\sqrt{v-t}}$$ \hspace{1cm} (34)

The corresponding Thetas, Deltas and Thetas of Deltas are

$$\sqrt{v-t}\Theta_t^c(t,v,k_c) = \frac{\sigma^e}{2} \phi(d(k_c,v-t)) - k_c \sqrt{v-t} \Phi(-d(k_c,v-t))$$ \hspace{1cm} (35)

$$\sqrt{v-t}\Theta_t^p(t,v,k_p) = k_p \sqrt{v-t} \Phi(d(k_p,v-t)) + \frac{\sigma^e}{2} \phi(d(k_p,v-t))$$ \hspace{1cm} (36)

$$\Delta^c_t(R_t; t,v,k_c) = \Phi(-e(R_t; k_c,t,v,s))$$ \hspace{1cm} (37)

$$\Delta^p_t(R_t; t,v,k_p) = -\Phi(e(R_t; k_p,t,v,s))$$ \hspace{1cm} (38)

$$\partial_s^c(R_t; t,v,k_c) = \phi(e(R_t; k_c,t,v,s)) \int_t^v (e(R_t; k_c,t,s)) dR_s$$ \hspace{1cm} (39)

$$\partial_s^p(R_t; t,v,k_p) = \phi(e(R_t; k_p,t,v,s)) \int_t^v (e(R_t; k_p,t,s)) dR_s$$ \hspace{1cm} (40)

The hedge fund market volatility coefficient is

$$\sigma_{t,v}^{h,e} = \kappa_{t,v}^{HM} \sigma^e \quad \text{where} \quad \kappa_{t,v}^{HM} \equiv \beta + \sqrt{v-t} \left( \gamma_c \Delta^c_t(R_t; t,v,k_c) + \gamma_p \Delta^p_t(R_t; t,v,k_p) \right)$$ \hspace{1cm} (41)

with \( \Delta^c_t(R_t; t,v,k_c) = 1_{\{R_t > k_c(t-v)\}} \) and \( \Delta^p_t(R_t; t,v,k_p) = -1_{\{R_t < k_p(t-v)\}} \). The implied market price of hedge fund risk is

$$\theta_{t,v}^{h,HM} = \theta_{t,v}^{h,S,HM} + \theta_{t,v}^{h,T,HM}$$ \hspace{1cm} (42)

where the selection component is \( \theta_{t,v}^{h,S,HM} = \alpha/\sigma^{h,p} \) and the timing component becomes

$$\theta_{t,v}^{h,T,HM} = \frac{\sqrt{v-t}}{\sigma^{h,p}} \left( \gamma_c \Theta_t^c(t,v,k_c) + \gamma_p \Theta_t^p(t,v,k_p) \right)$$

\[+ \frac{1}{2 \sqrt{v-t}} \left( \gamma_c C_t(t,v,k_c) + \gamma_p P_t(t,v,k_p) \right) \]

\[+ \frac{1}{2 \sigma^e \sigma^{h,p} \sqrt{v-t}} \int_t^v \left( \gamma_c \Delta^c_t(R_t; t,v,k_c) + \gamma_p \Delta^p_t(R_t; t,v,k_p) \right) dR_s \]

\[+ \frac{\sqrt{v-t}}{\sigma^e \sigma^{h,p}} \int_t^v \left( \gamma_c \partial_s^c(R_t; t,v,k_c) + \gamma_p \partial_s^p(R_t; t,v,k_p) \right) dR_s. \] \hspace{1cm} (43)

In the HM setting, the market volatility coefficient of the hedge fund return, \( \sigma_{t,v}^{h,e} \), remains proportional to the market volatility \( \sigma^e \) with proportionality factor \( \kappa_{t,v}^{HM} \). The proportionality factor is still positively related to systematic risk \( \beta_t \) and a function of timing ability \( (\gamma_c, \gamma_p) \), but it now depends nonlinearly on realized returns over the timing window. This return dependence arises through the Delta hedges of the option factors. When average returns \( R_{t,v}/(v-t) \) exceed the call threshold \( k_c \), volatility is an increasing function of the bull market timing ability \( \gamma_c \). When
average returns fall below the put threshold $k_p$, volatility is a decreasing function of the bear market timing ability. Hedge funds playing the extremes ($\gamma_c, \gamma_p > 0$ and $k_p \geq k_c$) exhibit increased (resp. reduced) volatility when average returns $R_{t,v}^e/(v - t)$ exceed $k_c$ (resp. fall below $k_p$). Hedge funds playing reverse straddles or strangles ($\gamma_c, \gamma_p < 0$ and $k_p \geq k_c$) exhibit reduced (resp. increased) volatility when average returns $R_{t,v}^e/(v - t)$ exceed $k_c$ (resp. fall below $k_p$). Market volatility is null at times when the intrinsic exposure to market risk is offset by the impact of timing ability (i.e., when $\beta + \sqrt{v - t} \left( \gamma_c \Delta^c_v(R_{t,v}^e; t, v, k_c) + \gamma_p \Delta^p_v(R_{t,v}^e; t, v, k_p) \right) = 0$). It is persistently null if the hedge fund is intrinsically market neutral ($\beta = 0$) and has no timing ability ($\gamma_c = \gamma_p = 0$).

The market price of hedge fund risk is path-dependent and non-Markovian. It depends on the selection ability coefficient ($\alpha$), the time elapsed since the beginning of the timing window $(v - t)$, the cumulative market excess return $(R_{t,v}^e)$ and its increments $(dR_{t,v}^e)$. It is well known that the Thetas of put and call options are non-monotone in the underlying price/factor. The market price of idiosyncratic hedge fund risk inherits this property. It also exhibits a non-monotone dependence on the length of the timing window.

The first two lines of (43) do not depend on realized returns. They capture the effects of the Thetas $\Theta^c_t(t, v, k_c), \Theta^p_t(t, v, k_p)$ and the timing option prices $C_t(t, v, k_c), P_t(t, v, k_p)$ on the timing component of the market price of risk. The remaining two lines of (43), which capture the effects of Deltas and Thetas of Deltas, depend on realized returns. As indicated above, this dependence is non-monotone: the timing component of the market price of risk can be increasing or decreasing in realized returns, depending on parameter values. The dependence on the length of the timing window is also non-monotone.

More can be said when the timing thresholds for excess returns are null ($k_c = k_p = 0$). In this case the first two terms reduce to the constant
\[
\frac{3}{2\sigma_{h,p}} (\gamma_c + \gamma_p) \frac{\sigma^e}{\sqrt{2\pi}}.
\] (44)

The next two components still display a dependence on realized returns. For funds playing the extremes (long straddles, i.e. $\gamma_c, \gamma_p > 0$) the market price of hedge fund risk is initially given by the constant in (44). After the initial date, it is subject to multiple and conflicting effects through Deltas and Thetas of Deltas. Sequences of positive returns (persistent bull markets) tend to be associated with positive Delta effects and negative Theta of Delta effects. But the latter are weak (strong) at the beginning (toward the end) of the timing window and therefore dominated (dominant). As a result persistent bull markets tend to be associated with positive and increasing market price of hedge fund risk early during the window, but decreasing and possibly negative later on. Sequences of negative returns (persistent bear markets) lead to the opposite patterns. When positive and negative returns alternate in random fashion, alternative and very diverse configurations can emerge.

The next corollary summarizes properties of components in the market price of hedge fund risk.
Corollary 4. The components in Corollary 3 have the following properties

(i) Timing option values are bounded and deterministic. For any \( n \in \mathbb{N} \),
\[
\lim_{v \uparrow t} \left( \frac{C_t(t, v, k_c)}{\sqrt{v - t}} \right)^n = (\sigma^e \phi(0))^n = \lim_{v \uparrow t} \left( \frac{P_t(t, v, k_p)}{\sqrt{v - t}} \right)^n
\]
(ii) Thetas are bounded and deterministic. For any \( n \in \mathbb{N} \),
\[
\lim_{v \uparrow t} \left( \sqrt{v - t} \Theta^e_t(t, v, k_c) \right)^n = \left( \frac{1}{2} \right)^n \lim_{v \uparrow t} \left( \frac{C_t(t, v, k_c)}{\sqrt{v - t}} \right)^n = \left( \frac{\sigma^e}{2} \phi(0) \right)^n
\]
\[
\lim_{v \uparrow t} \left( \sqrt{v - t} \Theta^p_t(t, v, k_p) \right)^n = \left( -\frac{1}{2} \right)^n \lim_{v \uparrow t} \left( \frac{P_t(t, v, k_p)}{\sqrt{v - t}} \right)^n = \left( \frac{\sigma^e}{2} \phi(0) \right)^n.
\]
(iii) Deltas are bounded and deterministic. For any \( n \in \mathbb{N} \),
\[
\lim_{s \uparrow v} \Delta^e_s(R^{e}_{t,s}; t, v, k_c) = 1 \{ R^{e}_{t,v} > k_c(v-t) \} \quad \text{and} \quad \lim_{s \uparrow v} \Delta^p_s(R^{e}_{t,s}; t, v, k_p) = -1 \{ R^{e}_{t,v} < k_p(v-t) \}
\]
\[
\lim_{s \downarrow v} \Delta^e_s(R^{e}_{t,s}; t, v, k_c) = \frac{1}{2} \quad \text{and} \quad \lim_{s \downarrow v} \Delta^p_s(R^{e}_{t,s}; t, v, k_p) = -\frac{1}{2}.
\]
The process \( \left( \sqrt{v - t} \right)^{-1} \int_t^v \left( \gamma_e \Delta^e_s(R^{e}_{t,s}; t, v, k_c) + \gamma_p \Delta^p_s(R^{e}_{t,s}; t, v, k_p) \right) dR^e_s \) is a martingale.

(iv) Thetas of Deltas are stochastic and satisfy
\[
\lim_{s \uparrow v} (\sqrt{v - t} \theta^e_s(t, v, k)) = 0 \quad \text{and} \quad \lim_{s \downarrow v} (\sqrt{v - t} \theta^e_s(t, v, k)) = -\phi(0) \frac{k}{2}.
\]

The limits in Corollary 4 show that the scaling factor \( g(\tau - t) \equiv \sqrt{\tau - t} \) in the extended HM model is necessary to ensure a non-explosive market price of risk (\( \Theta^e_t(t, v, k_c) \) goes to infinity, but \( \sqrt{v - t} \Theta^e_t(t, v, k_c) \) converges to a constant as \( v - t \to 0 \)). This precludes arbitrage opportunities and ensures that the asset allocation problem of an outside investor is well-posed. Thus, the market price of hedge fund risk defines an equivalent martingale measure.

Corollary 5. Consider the economy with the market portfolio and the hedge fund with timing window \([t, \tau]\). Assume that the market price of equity risk \( \theta^e \) is constant. The unique martingale measure \( Q \) for this economy has density process \( Z^{e}_{t,v} = Z^{h}_{t,v}Z_{t,v} \), where
\[
Z^{e}_{t,v} = \exp \left( -\theta^e (W^e_v - W^e_t) - \frac{1}{2} (\theta^e)^2 (v - t) \right)
\]
\[
Z^{h}_{t,v} = \exp \left( -\int_t^v \left( \theta^h_{t,s} \right)^{HM} \right) dW^p_s - \frac{1}{2} \int_t^v \left( \theta^h_{t,s} \right)^{HM)^2 ds \right) .
\]
The measure \( Q \) is equivalent to \( P \), i.e., \( P_t(Z_{t,v} = 0) = P_t \left( \int_t^v \left( \theta^h_{t,s} \right)^{HM)^2 ds = +\infty \right) = 0. \)

Corollary 5 establishes the existence of an equivalent martingale measure relative to public information, over the length of the timing window. Under these conditions, the portfolio choice problem of an outside investor becomes meaningful.
4 Optimal Portfolios

This section examines the optimal portfolio policy in the presence of market-timing hedge funds. The presentation focuses on constant relative risk aversion. Section 4.1 considers unrestricted portfolios. Section 4.2 discusses the impact of short sales constraints. The implications of additional traded and non-traded factors are examined in Section 4.3.

4.1 Unconstrained Portfolios

4.1.1 Optimal Unconstrained Portfolio in the TM Model

The optimal portfolio when the hedge fund strategies are captured by the TM model is described in the next proposition.

**Proposition 2.** Suppose that the hedge fund return is described by the TM model and let \([t, \tau]\) be the market timing window. Consider an individual with constant relative risk aversion \(R\) and investment horizon \(T\). The unconstrained optimal portfolio (the fraction of wealth invested) is \(\pi_v = \pi_v^m\) where \(\pi_v^m\) is the mean-variance demand

\[
\pi_v^m \equiv \begin{bmatrix}
\frac{1}{\sigma^e} & -\frac{\sigma_h^e}{\sigma^e \sigma^{h,v}} \\
0 & \frac{1}{\sigma^{h,v}}
\end{bmatrix}
\begin{bmatrix}
\theta^e \\
\theta^h_{t,v}
\end{bmatrix} = \frac{1}{R} \begin{bmatrix}
\frac{\theta^e}{\sigma^e} - \kappa_{t,v} T M \frac{\alpha + \gamma T M (\sigma^e)^2}{(\sigma^{h,v})^2} \\
\frac{\alpha + \gamma T M (\sigma^e)^2}{(\sigma^{h,v})^2}
\end{bmatrix}
\] (45)

for \(v \in [t, \tau]\). The portfolio policy is independent of the investment horizon \(T\).

When hedge fund returns have the TM structure and market coefficients are constant the optimal portfolio has the classic mean-variance structure. In this instance, the portfolio composition is entirely driven by the diversification motive.

Selection (\(\alpha T M\)) and timing (\(\gamma T M\)) abilities of the fund manager affect the demand for the hedge fund and the demand for stocks. The impact on the demand for the hedge fund is clear. Selection and timing abilities determine the premium associated with idiosyncratic hedge fund risk (the price of hedge fund risk \(\theta^h_{t,v} = (\alpha + \gamma T M (\sigma^e)^2) / \sigma^{h,v}\)), which in turn drives the demand for the fund. When timing and selection abilities offset each other (\(\alpha + \gamma T M (\sigma^e)^2 = 0\)) the hedge fund is a redundant asset, which will not be held. In contrast, a positive (resp. negative) reward \(\alpha + \gamma T M (\sigma^e)^2 > 0\) (resp. \(\alpha + \gamma T M (\sigma^e)^2 < 0\)) induces a long (resp. short) position in the hedge fund. If selection and timing abilities increase, the reward for hedge fund risk increases, driving up the demand for investment in the fund. The impact of selection and timing on the demand for stocks is more subtle, as it is driven by the correlation between hedge fund return and stock market return (\(\theta^T M_{t,v}\)). The investment in the hedge fund creates an exposure to stock market risk. The demand for stocks is modified by the extent of this exposure.
The specific impact on the demand for stocks is determined by the sign of the correlation \( \rho_{T M}^{t,v} \), which is the same as the sign of

\[
\kappa_{T M}^{T M} = \beta_{T M} + 2\gamma_{T M} R_{t,v}^e.
\]

This follows from Proposition 1, which shows that the exposure of hedge fund returns to market risk is determined by the volatility coefficient \( \sigma_{T M}^{h,e} = \kappa_{T M}^{T M} \sigma_e^e \). The interesting aspect here is that correlation is directly related to timing ability and to the realized market returns up to the decision point of the investor. When timing ability and realized market returns are such that \( \kappa_{T M}^{T M} = 0 \) there is no correlation between hedge fund and market returns implying that stocks are useless in hedging market exposures due to hedge fund investments. The demand for stocks is then independent of hedge fund characteristics (such as \( \alpha, \gamma_{T M} \)).

When \( \kappa_{T M}^{T M} > 0 \) (resp. \( \kappa_{T M}^{T M} < 0 \)) a positive exposure to market risk through the amount invested in the hedge fund, which arises when the reward \( \alpha + \gamma_{T M} (\sigma^e)^2 \) is positive, is offset by a reduction (resp. increase) in the demand for stocks. It is also interesting to note that, in the special case \( \kappa_{T M}^{T M} = 1 \), the demand for cash \( 1 - \pi_v = 1 - \theta^e / (\sigma^e R) \) becomes insensitive to the hedge fund characteristics \( (\alpha, \gamma_{T M}, \sigma_{h,p}) \) (i.e., the presence of the hedge fund does not affect the fraction in cash).

The effects of various parameters can now be clearly understood. For instance, a higher stock market volatility \( (\sigma^e) \) enhances the usefulness of timing ability and therefore the reward for risk. This increases the demand for the hedge fund and reduces (increases) that for stocks when \( \kappa_{T M}^{T M} > 0 \) (\( \kappa_{T M}^{T M} < 0 \)). A higher stock market volatility also reduces the reward for stock market risk, causing a direct reduction in the fraction invested in stocks. The overall impact is the combination of these two effects. Likewise, an increase in hedge fund idiosyncratic risk \( (\sigma_{h,p}) \) reduces the attractiveness of the fund leading to a reduced hedge fund demand and an increased (reduced) stock demand when \( \kappa_{T M}^{T M} > 0 \) (\( \kappa_{T M}^{T M} < 0 \)).

### 4.1.2 Optimal Unconstrained Portfolio in the HM Model

In the HM model, the market price of hedge fund risk is path-dependent. Timing induces an additional intertemporal hedging demand, as shown next.

**Proposition 3.** Suppose that the hedge fund return is described by the HM model and let \([t, \tau]\) be the market timing window. Consider an individual with constant relative risk aversion \( R \) and investment horizon \( T \). The unconstrained optimal portfolio (the fraction of wealth invested) is \( \pi_v = \pi_v^m + \pi_v^\theta \) where \( \pi_v^m \) is the mean-variance demand

\[
\pi_v^m = \frac{1}{R} \begin{bmatrix}
\frac{1}{\sigma^e} - \kappa_{T M}^{T M} \theta_{h,v}^{T M} \\
0
\end{bmatrix} \begin{bmatrix}
\theta^e \\
\theta_{t,v}^h
\end{bmatrix} = \frac{1}{R} \begin{bmatrix}
\frac{\theta^e}{\sigma^e} - \kappa_{T M}^{T M} \theta_{h,v}^{T M} \\
\frac{\theta_{t,v}^h}{\sigma_{h,p}^v}
\end{bmatrix}
\]

(46)
and where $\pi_v^\theta$ is a (dynamic) market price of risk hedge

$$
\pi_v^\theta = \frac{\rho (\rho - 1)}{\sigma_v^e} \left[ E \left[ \frac{V_{v,T}}{E[V_{v,T} | F^e_{v}]} \int_{v}^{\tau \land T} \theta_t^h \theta_t^h d\theta_t^h ds \right] \right]_{F^e_v} 
$$

with $\rho = 1 - 1/R$ and

$$
V_{v,T} = \exp \left( -\rho \theta^e(We_{\tau \land T} - We_v) + \frac{\rho (\rho - 1)}{2} \int_{v}^{\tau \land T} (\theta_t^h)^2 ds \right),
$$

The expression for the Malliavin derivative $D_v^e \theta_t^h$ is given in (72)-(75) in Appendix B. The portfolio policy depends on the investment horizon $T$ through the market price of risk hedge.

With HM strategies the hedge fund premium becomes a function of the realized market return (see Corollary 3). This dependence induces a radical change in the optimal portfolio. Any position in the hedge fund now creates an exposure to market risk, not only because the hedge fund is partly invested in the market, but also because the reward attached to that investment depends on market returns. The result is the emergence of a dynamic hedging demand (47) which complements the mean-variance demand (46). This dynamic hedging motive only affects the demand for stocks: the investment in the hedge fund is purely motivated by risk-reward considerations. Stocks, in contrast, can be used to hedge the exposure to market risk created by the hedge fund position.

The structure of the reward for hedge fund risk sheds new light on the elements underlying the diversification-driven hedge fund demand. As discussed previously, this reward has a tendency to initially increase (decrease) then decrease (increase) in a persistent bull (bear) market. The investment in the hedge fund reflects this pattern: during a persistent bull (bear) market it rises (falls) initially, then falls (rises) during the timing window.

Equities are used, in part, to hedge the risk exposure implied by the hedge fund position. For this reason, the mean-variance component of the demand for equities (the first line of (46)) reflects the investment pattern in the hedge fund. At the start of a persistent bull (bear) market that component reduces (increases) the demand for equities. At later stages of the timing window, it raises (lowers) the demand for equities. The basic effect, due to the behavior of the market price of hedge fund risk, is enhanced (tamed) during persistent bull (bear) markets by the increased (reduced) volatility of hedge fund returns and correlation with equities returns. Inversions of the behavior described can even take place in bear markets, if volatility and correlation become negative.

The intertemporal fluctuations in the market price of hedge fund risk also generate the dynamic hedging motive (47) in the demand for equities. This component reflects the anticipated evolution of the market price of risk and its sensitivity to market shocks. The overall behavior and evolution during the timing window is difficult to assess. The numerical study carried out later on in the paper, documents some of its properties.
4.2 No Short Sales Constraint

In practice, short positions in hedge funds cannot be implemented. This section studies the impact of such a short sales constraint on the optimal portfolio policy.

For hedge funds with returns described by the TM model, the optimal constrained portfolio is

**Proposition 4.** Suppose that hedge funds cannot be held short. Also suppose that the hedge fund return is described by the TM model and let \([t, \tau]\) be the market timing window. Consider an individual with constant relative risk aversion \(R\) and investment horizon \(T\). The (no short sales) constrained optimal portfolio is \(\pi^v = \pi^m_v\) where

\[
\pi^m_v = \frac{1}{R} \left[ \frac{\theta_{e} \sigma^2 - \kappa_{M}^{T} \frac{(\alpha + \gamma^{T} \sigma^2)^+}{(\sigma^2)^+}}{\kappa_{M}^{T} (\sigma^2)^+} \right].
\]

(49)

with \((x)^+ \equiv \max (x, 0)\).

As in the unconstrained case, the optimal portfolio remains driven by risk-reward considerations. The constraint first impinges of the amount allocated to the hedge fund. When the reward for hedge fund risk is positive \((\alpha + \gamma^{T} \sigma^2 > 0)\) the investor optimally takes a long position in the fund and behaves as in the unconstrained case. When the reward is negative \((\alpha + \gamma^{T} \sigma^2 < 0)\) the investor would like to take a short position, but cannot due to the constraint. The optimal choice is then not to hold the hedge fund. As before, the position in the hedge fund induces an exposure to market risk, which motivates an adjustment in the demand for stocks. When the constraint binds the investment in the hedge fund is null and the need to hedge this exposure vanishes. This is the second effect of the no short sales constraint.

To provide additional perspective it is useful to recall that the reward for hedge fund risk, in the general case, is

\[
\sigma^{h,p} \theta^{h}_{t} = \alpha (v - t) + \gamma \left( \Theta_{t} (t, v) + \int_{t}^{v} \vartheta_{s} (t, v) dR_{s} \right).
\]

When this reward is negative the constraint binds and the demand for the hedge fund investment will be null. This can happen because \(\sigma^{h,p} > 0\) and \(\theta^{h}_{t} < 0\) or because \(\sigma^{h,p} < 0\) and \(\theta^{h}_{t} > 0\). In the TM timing model it occurs when \(\alpha + \gamma^{T} \sigma^2 < 0\).

The corresponding result for the HM timing model is as follows

**Proposition 5.** Suppose that hedge funds cannot be held short. Also suppose that the hedge fund return is described by the HM model and let \([t, \tau]\) be the market timing window. Consider an individual with constant relative risk aversion \(R\) and investment horizon \(T\). The (no short sales) constrained optimal portfolio is \(\pi^v = \pi^m_v + \pi^{\theta}_v\) with mean-variance and market price of risk hedge
components given by

\[
\pi^m_v = \frac{1}{R} \left[ \frac{v}{\sigma^e} - \theta_{t,v}^{HM} \left( \frac{(\sigma_{h,p}^h)^+}{(\sigma_{h,p})^+} \right) - \frac{(\sigma_{h,p}^h)^+}{(\sigma_{h,p})^+} \right]
\]  

(50)

\[
\pi^\theta_v = \rho \left( \rho - 1 \right) \left[ \frac{v}{\sigma^e} \right] \left[ \frac{E}{v_v} V_{v,T}^e \int_{v_v}^{\tau_T} \theta_{t,s}^h \Delta_t^e \theta_{t,s}^h 1_{\{\sigma_{h,p}^h \geq 0\}} ds \mid F_v^e \right]  
\]  

(51)

where

\[
V_{v,T}^e \equiv \exp \left( -\rho \theta^e (W_{T_A}^e - W_v^e) + \frac{\rho (\rho - 1)}{2} \int_{v_v}^{\tau_T} (\theta_{t,s}^h)^2 ds \right).
\]  

(52)

In these expressions \( \theta_{t,v}^h \) is given by the formulas in Corollary 3 and \( \theta_{t,v}^{h,e} = (\theta_{t,v}^h)^+ 1_{\{\sigma_{h,p}^h \geq 0\}} - (\theta_{t,v}^h)^- 1_{\{\sigma_{h,p}^h < 0\}} \).

In the HM model the portfolio demand has a mean-variance part and a dynamic hedging part. This remains true when the hedge fund cannot be held short. Moreover, the constrained mean-variance term \( \pi^m_v \) reflects the same type of incentives as in the TM model, albeit with a different market price of hedge fund risk \( \theta_{t,v}^h \). When the reward for hedge fund risk is negative (\( \sigma_{t,v}^{h,p} \theta_{t,v}^h < 0 \)) the investor wishes to take a short position in the fund, which is not possible. The mean-variance demand (which is also the total demand) for the hedge fund is then null (\( \pi_v^{m,h} = 0 \)) and the mean-variance demand for stocks is entirely driven by the reward for holding stocks (\( \pi_v^{m,e} = \theta^e / (R \sigma^e) \)).

The hedging demand \( \pi^\theta_v \), which is specific to the HM model, reflects the impact of the constraint. At times at which the constraint binds there is no investment in the hedge fund, hence no exposure to market risk. As a result there is no need to hedge fluctuations in the market price of hedge fund risk in those events. The need to hedge is confined to events \( \{\sigma_{t,v}^{h,p} \theta_{t,v}^h \geq 0\} \) in which the reward for hedge fund risk is nonnegative and holdings of the fund are nonnegative. The dynamic hedging demand has the same structure as in the unconstrained case except that it is adjusted for these events (see the indicator function in (51)).

### 4.3 Additional Risk Factors

This section examines the timing model and the associated optimal portfolio when additional factors are added to the regression equation. Motivation for this extension comes from the literature which has examined the relevance of certain factors. Hasanahodzic and Lo (2005), for instance, consider an additional bond factor, a credit factor, a commodity factor and an FX factor. Bilio et al. (2006) add other factors, like the Fama-French factors. In what follows, a specification of returns with \( p - 1 \) additional factors is considered. The cases where factors are traded portfolios and where they are nontraded will both be considered.
The return model becomes
\[
\begin{bmatrix}
    dR^e_v \\
    dR^f_v
\end{bmatrix} = \Sigma \left( \begin{bmatrix}
    \theta^e \\
    \theta^f
\end{bmatrix} dv + \begin{bmatrix}
    dW^e_v \\
    dW^f_v
\end{bmatrix} \right)
\] (53)
where \(dR^f_v\) is the vector of factor growth rates, \(\theta^f\) is the vector of prices of risk associated with the factor shocks \(W^f\) and \(\Sigma\) is a matrix of volatility coefficients. The Brownian motion \(W^f\) representing pure factor shocks has dimension \(p - 1\). The volatility matrix can be partitioned as
\[
\Sigma = \begin{bmatrix}
    \sigma^e & 0 \\
    \Sigma_{ef} & \Sigma_{ff}
\end{bmatrix}
\]
so that \(W^e\) represents equity market risk. Factor growth rates are therefore affected by equity market shocks (because of \(\Sigma_{ef}\)) and pure factor shocks (due to \(\Sigma_{ff}\)). The matrix \(\Sigma_{ef}\) is the covariance between equity returns and factor growth rates. In the spirit of the previous sections the coefficients \(\Sigma, \theta^e, \theta^f\) are assumed to be constant.

The model for hedge fund returns becomes
\[
dR^h_{t,v} = \pi^h,e V_t dR^e_v + \pi^h,f V_t dR^f_v + \pi^h,p V_t dR^p_v
\] (54)
\[
= \sigma^h,e (\theta^e dv + dW^e_v) + \sigma^h,f (\theta^f dv + dW^f_v) + \sigma^h,p (\theta^h_{t,v} dv + d\tilde{W}^p_v) \tag{55}
\]
where the second line is obtained by using the same arguments as before. The extended regression model is
\[
R^h_{t,\tau} = \mathbb{E}_t \left[ R^h_{t,\tau} \mid \mathcal{F}^e_{\tau, f} \right] + \left( R^h_{t,\tau} - \mathbb{E}_t \left[ R^h_{t,\tau} \mid \mathcal{F}^e_{\tau, f} \right] \right)
\]
\[
= \alpha (\tau - t) + \beta \int_t^\tau dR^e_v + \gamma f (t, \tau, R^h_{t,\tau}) + \delta^f \int_t^\tau dR^f_v + \sigma^h,p \int_t^\tau d\tilde{W}^p_v \tag{56}
\]
where \(\delta^f\) is a \(p - 1\) dimensional vector of factor loadings associated with the \(p - 1\) additional factors. Identification of the coefficients in (55) and (56) gives the volatility structure
\[
(\sigma^h_{t,v})' = \left( \sigma^h,e_{t,v} + \delta^e \Sigma_{ef}, \delta^f \Sigma_{ff}, \sigma^h,p \right)
\]
where \(\sigma^h,e, \sigma^h,p\) are the volatility coefficient with respect to \(W^e\) and \(\tilde{W}^p\) in the model without factors and \(\delta^e \Sigma_{ef}, \delta^f \Sigma_{ff}\) are the volatility modifications due to the additional factors. The volatility \(\delta^e \Sigma_{ef}\) captures the exposure of hedge fund returns to pure factor risks \(W^f\). The term \(\delta^f \Sigma_{ff}\) is the incremental exposure to market risk \(W^e\) due to the sensitivity of factor returns to market shocks.

### 4.3.1 Traded Risk Factors

Suppose first that the \(p - 1\) additional risk factors are investable portfolios. In this case the volatility matrix of traded assets returns is
\[
\sigma_{t,v} = \begin{bmatrix}
    \sigma^e & 0' & 0 \\
    \Sigma_{ef} & \Sigma_{ff} & 0 \\
    \sigma^h,e_{t,v} + \delta^e \Sigma_{ef} & \delta^e \Sigma_{ff} & \sigma^h,p
\end{bmatrix} \tag{57}
\]
Optimal portfolios for TM hedge fund strategies and HM strategies are given in the next two propositions. Both the unconstrained and the no short sales constraint cases are covered.

**Proposition 6.** Suppose that the hedge fund return is described by the TM model with additional traded risk factors and let \([t, \tau]\) be the market timing window. Consider an individual with constant relative risk aversion \(R\) and investment horizon \(T\). The optimal unconstrained portfolio is \(\pi_v = \pi_v^m\) with mean-variance component given by

\[
\pi_v^m = \frac{1}{R} \left[ \begin{array}{c} \bar{\sigma}^e - \frac{(\Sigma_{e1})'\Sigma^{-1}f}{\sigma^e} \delta f - \kappa_{TM} \theta_{t,v}^{h, TM} \theta_{t,v}^{h, p} \\ \sum_{jj}^{-1} \theta^f - \delta \theta_{t,v}^{h, TM} \\ \theta_{t,v}^{h, TM} \sigma_{h, p} \end{array} \right] \tag{58}
\]

where \(\theta_{t,v}^{h, TM}\) is defined in (42). The optimal portfolio weights with no short sales constraint are obtained by replacing \(\theta_{t,v}^{h, TM}\) with \(\theta_{t,v}^{h, TM,c} = (\theta_{t,v}^{h, TM})^+ \mathbb{1}_{\{\sigma_{h, p} \geq 0\}} - (\theta_{t,v}^{h, TM})^- \mathbb{1}_{\{\sigma_{h, p} < 0\}}\) in (58).

**Proposition 7.** Suppose that the hedge fund return is described by the HM model with additional traded risk factors and let \([t, \tau]\) be the market timing window. Consider an individual with constant relative risk aversion \(R\) and investment horizon \(T\). The optimal unconstrained portfolio is \(\pi_v = \pi_v^m + \pi_v^\theta\) with mean-variance and market price of risk hedge components given by

\[
\pi_v^m = \frac{1}{R} \left[ \begin{array}{c} \bar{\sigma}^e - \frac{(\Sigma_{e1})'\Sigma^{-1}f}{\sigma^e} \delta f - \kappa_{HM} \theta_{t,v}^{h, HM} \theta_{t,v}^{h, p} \\ \sum_{ff}^{-1} \theta^f - \delta \theta_{t,v}^{h, HM} \\ \theta_{t,v}^{h, HM} \sigma_{h, p} \end{array} \right] \tag{59}
\]

\[
\pi_v^\theta = \frac{\rho(\rho - 1)}{\sigma^e} \left[ \mathbb{E} \left[ \sum_{t,s}^T \theta_{t,s}^{h, HM} D_{t,s}^e \theta_{t,s}^{h, HM} ds \Bigg| \mathcal{F}_v^e \right] \right] \tag{60}
\]

where \(V_{c,T}\) is as defined in (48). The optimal portfolio weights with no short sales constraint are obtained by replacing \(\theta_{t,v}^{h, HM}\) with \(\theta_{t,v}^{h, HM,c} = (\theta_{t,v}^{h, HM})^+ \mathbb{1}_{\{\sigma_{h, p} \geq 0\}} - (\theta_{t,v}^{h, HM})^- \mathbb{1}_{\{\sigma_{h, p} < 0\}}\) in (60).

### 4.3.2 Non-traded Risk Factors

When the additional risk factors are nontraded, the financial market is incomplete. The shadow price of nontraded risk must then be identified along with the optimal policy. The next proposition presents the solution for TM hedge fund strategies.

**Proposition 8.** Suppose that the hedge fund return is described by the TM model with additional nontraded risk factors and let \([t, \tau]\) be the market timing window. Consider an individual with
constant relative risk aversion $R$ and investment horizon $T$. The optimal unconstrained portfolio is $\pi_t = \pi^m_t$ with mean-variance component given by

$$
\pi^m_t = \frac{1}{R} \left[ \frac{\theta^e}{\sigma^e} - \left( \kappa^{TM}_t + \frac{\sum_{j} d_j}{\sigma^e} \right) \frac{\phi^{TM}_t}{\sigma^e} \theta^{h,TM}_t \right]
$$

(61)

where $\phi^{h,TM}_t$ is defined in (42). The optimal portfolio weights with a no short sales constraint are obtained by replacing $\theta^{h,TM}_t$ with $\theta^{h,TM,c}_t = \left( \theta^{h,TM}_t \right)^+ \mathbb{1}_{\{\sigma^h,p \geq 0\}} - \left( \theta^{h,TM}_t \right)^- \mathbb{1}_{\{\sigma^h,p < 0\}}$ in (58).

The main feature of the solution is that the (vector) shadow price associated with the nontraded risk factors is deterministic. The demand for the hedge fund remains driven by the risk-reward trade-off associated with unique hedge fund risk. That risk-reward trade-off is not affected by the presence of nontraded risk factors, leading to the same mean-variance demand as in the model with traded factors. Taking a position in the hedge fund now generates two distinct exposures, one to market risk, the other to nontraded risk. The endogenous shadow price of nontraded risk ensures that the investor is content to live with this nontraded exposure. This shadow price can be retrieved directly from the portfolio in Proposition 6. It equals

$$
\theta^f = \sum_{fJ} \frac{\phi^{h,HM}_t}{\sigma^h,p}
$$

(set the demand for investment in factors equal to zero and solve for the implied shadow price $\theta^f$).

Substituting back in the demand for stocks in the same proposition now gives the stock demand in the nontraded environment. Given that $\theta^f$ is proportional to the market price of hedge fund risk, the resulting stock demand has the structure announced. The impact of nontradedness is captured by the term $-\left( \sum_{ef} \delta/\sigma^e \right) \phi^{h,TM}_t \frac{\theta^{h,HM}_t}{\sigma^h,p}$. When the market price of hedge fund risk per unit risk is positive, the investor finds it optimal to reduce (increase) the allocation to stocks if $(\sum_{ef})^{\delta/\sigma^e} > 0$ (resp. $(\sum_{ef})^{\delta/\sigma^e} < 0$).

For HM hedge fund strategies, optimal portfolios become

**Proposition 9.** Suppose that the hedge fund return is described by the HM model with additional traded risk factors and let $[t, \tau]$ be the market timing window. Consider an individual with constant relative risk aversion $R$ and investment horizon $T$. The optimal unconstrained portfolio is $\pi_v = \pi^m_v + \pi^\theta_v$ with mean-variance and market price of risk hedge components given by

$$
\pi^m_v = \frac{1}{R} \left[ \frac{\theta^e}{\sigma^e} - \left( \kappa^{H,M}_t + \frac{\sum_{j} d_j}{\sigma^e} \right) \frac{\phi^{H,M}_t}{\sigma^e} \theta^{h,HM}_t \right]
$$

(62)

$$
\pi^\theta_v = (1 + \eta^f) \rho (\rho - 1) \left[ \mathbb{E} \left[ \frac{V^n_{t,T}}{V^n_{v,T}} | F_v \right] \int_{\tau}^{T} \phi^{h,HM}_t \theta^{h,HM}_t D_v \theta^{h,HM}_t ds \bigg| F_v \right]
$$

(63)
where $\eta^f \equiv \frac{\delta \Sigma f \Sigma \delta}{(\sigma^h,p)}$ and

$$V^n_{V,T} \equiv \exp \left( -\rho^e (W^e_{\tau,T} - W^e_v) + (1 + \eta^f) \frac{\rho(\rho - 1)}{2} \int_v^{\tau,T} (\theta^h_{t,s}^{HM})^2 ds \right).$$  \hspace{1cm} (64)

The optimal portfolio weights with a no short sales constraint are obtained by replacing $\theta^h_{t,v}^{HM}$ with $\theta^h_{t,v}^{HM,c} = (\theta^h_{t,v}^{HM})^+ \{ \sigma^h,p \geq 0 \} - (\theta^h_{t,v}^{HM})^- \{ \sigma^h,p < 0 \}$ in (60).

With HM hedge fund strategies the shadow price of uncertainty remains of the form $\theta^I = \Sigma_f \delta \theta^h_{t,v}^{HM}$. But now the market price $\theta^h_{t,v}^{HM}$ has the nonlinear structure described earlier, leading to a new hedging component in the demand for stocks. This hedging component is again driven by the return-dependence of $\theta^h_{t,v}^{HM}$, but its origin is the endogenous price of nontradedness. The demand for stocks has now two hedging components. The first is the same as before, motivated by fluctuations in the price of nontradedness. The second is new and driven by fluctuations in the endogenous price of nontradedness (this is the term with $\eta^f$ in (63)). The two terms are proportional to each other, because they are both related to the structure of $\theta^h_{t,v}^{HM}$.

5 Empirical Implementation

Hedge fund holding period returns are typically reported at a monthly frequency. In order to implement the previous analysis, these holding period returns must first be converted to continuously compounded excess returns. This is accomplished by using discounted prices

$$\frac{b_{t+1} S^h_{t+1}}{b_t S^h_{t}} = \frac{1 + r^h_{t+1}}{1 + r^h_{t+1}} \quad \text{and} \quad \frac{b_{t+1} S^e_{t+1}}{b_t S^e_{t}} = \frac{1 + r^e_{t+1}}{1 + r^e_{t+1}}$$

where $b_t = \exp \left( -\int_0^t r_s ds \right)$ and $r^h_{t+1}, r^e_{t+1}$ are the observed hedge fund, equity and riskless returns between $t_i$ and $t_{i+1}$ (holding period returns). The passage to the continuously compounded rates in the models uses the relations

$$\int_{t_i}^{t_{i+1}} dR^h_v = \log \left( \frac{b_{t_{i+1}} S^h_{t_{i+1}}}{b_{t} S^h_{t_{i}}} \right) + \frac{1}{2} \int_{t_i}^{t_{i+1}} \left( \sigma^h_{t_i,v}^2 + (\sigma^h_{t_i,v})^2 \right) dv$$ \hspace{1cm} (65)

$$\int_{t_i}^{t_{i+1}} dR^e_v = \log \left( \frac{b_{t_{i+1}} S^e_{t_{i+1}}}{b_{t} S^e_{t_{i}}} \right) + \frac{1}{2} \int_{t_i}^{t_{i+1}} (\sigma^e_{v})^2 dv.$$ \hspace{1cm} (66)

(These follow from Ito’s lemma), where $R^h$ on the left hand side of (65) is given by the timing model (14). The rest of the implementation depends on the specific timing strategies adopted by the hedge fund. As will be seen shortly, stock market information can and will be used at a much finer frequency than the hedge fund reporting frequency in order to estimate hedge fund return characteristics.\hspace{1cm} \footnote{The window used to construct stock returns in (65) is arbitrary and does not need to match that for hedge funds.}
5.1 Implementation for TM Strategies

In the TM model with independent increments, the regression equation (65)-(66), (14) becomes

\[
\log \left( \frac{1 + r_{t_i+1}^h}{1 + r_{t_i+1}^f} \right) = -\frac{1}{2} \int_{t_i}^{t_{i+1}} (\sigma_{t_i,v}^{h,e})^2 \, dv + a\Delta + b \log \left( \frac{1 + r_{t_i+1}^e}{1 + r_{t_i+1}^f} \right) + \gamma_{TM} \left( \log \left( \frac{1 + r_{t_i+1}^e}{1 + r_{t_i+1}^f} \right) \right)^2 + \sigma_{h,p} \left( \tilde{W}_{t_{i+1}}^p - \tilde{W}_{t_i}^p \right) \tag{67}
\]

where \( \Delta = t_{i+1} - t_i \) is the constant reporting interval and

\[
a \equiv \alpha + \frac{1}{2} \beta (\sigma^e)^2 + \frac{1}{4} \gamma_{TM} (\sigma^e)^4 \Delta - \frac{1}{2} (\sigma_{h,p})^2, \quad b \equiv \beta + \gamma_{TM} (\sigma^e)^2 \Delta \tag{68}
\]

\[
\frac{1}{2} \int_{t_i}^{t_{i+1}} (\sigma_{t_i,v}^{h,e})^2 \, dv = \frac{\beta^2}{2} (\sigma^e)^2 \Delta + \beta \frac{\gamma_{TM} (\sigma^e)^4 \Delta^2}{2} + \frac{1}{6} (\sigma^e)^6 (\gamma_{TM})^2 \Delta^3
\]

\[
+ 2 \beta \gamma_{TM} (\sigma^e)^2 \int_{t_i}^{t_{i+1}} \left( \frac{1 + r_v^e}{1 + r_v^f} \right) \, dv
\]

\[
+ 2 (\gamma_{TM})^2 (\sigma^e)^4 \int_{t_i}^{t_{i+1}} \left( \frac{1 + r_v^e}{1 + r_v^f} \right) (v - t_i) \, dv
\]

\[
+ 2 (\gamma_{TM})^2 (\sigma^e)^2 \int_{t_i}^{t_{i+1}} \left( \frac{1 + r_v^e}{1 + r_v^f} \right)^2 \, dv \tag{69}
\]

where \( r_v^e, r_v^f \) are the holding period returns over the interval \([t_i, u]\) (see proof in Appendix D). Equations (67)-(69) constitute the TM regression model. The last component, (69), shows that equity returns at a higher frequency than \( \Delta \) are useful for estimation purposes. Moreover, (67)-(69) can be estimated by a simple linear regression with nonlinear restrictions on parameters. This follows from the fact that the stochastic variables in (69) are additive and can therefore be added linearly to the regression model (67). Details are in Appendix C.

5.1.1 Implementation for HM Strategies

The same transformations give, for the HM model with homoscedastic residuals,

\[
\log \left( \frac{1 + r_{t_i+1}^h}{1 + r_{t_i+1}^f} \right) = -\frac{1}{2} \int_{t_i}^{t_{i+1}} (\sigma_{t_i,v}^{h,e})^2 \, dv + \left( \alpha - \frac{1}{2} (\sigma_{h,p})^2 + \frac{1}{2} \beta (\sigma^e)^2 \right) \Delta + \beta \log \left( \frac{1 + r_{t_i+1}^e}{1 + r_{t_i+1}^f} \right)
\]

\[
+ \gamma_p \sqrt{\Delta} \left( k_p - \frac{1}{2} (\sigma^e)^2 \right) \Delta - \log \left( \frac{1 + r_{t_i+1}^e}{1 + r_{t_i+1}^f} \right)^{+} + \gamma_c \sqrt{\Delta} \left( \log \left( \frac{1 + r_{t_i+1}^e}{1 + r_{t_i+1}^f} \right) - \left( k_c - \frac{1}{2} (\sigma^e)^2 \right) \Delta \right)^{+} + \sigma_{h,p} \left( \tilde{W}_{t_{i+1}}^p - \tilde{W}_{t_i}^p \right) \tag{70}
\]
where
\[-\frac{1}{2} \int_{t_i}^{t_{i+1}} \left( \sigma_{t_i,v}^{h,e} \right)^2 dv = -\frac{\beta^2 (\sigma^e)^2}{2} \Delta - \beta (\sigma^e)^2 \int_{t_i}^{t_{i+1}} \sqrt{v-t_i} \left( \gamma_c 1 \{ R_{t_i,v}^{c} > k_c(v-t_i) \} + \gamma_p 1 \{ R_{t_i,v}^{c} < k_p(v-t_i) \} \right) dv \]
\[-\frac{(\sigma^e)^2}{2} \int_{t_i}^{t_{i+1}} (v-t_i) \left( \gamma_c 1 \{ R_{t_i,v}^{c} > k_c(v-t_i) \} + \gamma_p 1 \{ k_p(v-t_i) > R_{t_i,v}^{c} \} \right)^2 d\tilde{v} \] (71)

If equity returns are available at a higher frequency than \( \Delta \), the random variables in the indicator functions can be calculated more explicitly (in fact very precisely with ultra-high frequency data), provided \( k_c \) and \( k_p \) are known. It follows that, given \( \sigma^e, k_c \), and \( k_p \), the other parameters \( \alpha, \beta, \gamma_c, \gamma_p, \sigma^{h,p} \) can be estimated using a linear nine factor regression model
\[ Y_{t_{i+1}} = \sum_{k=0}^{8} b_k X_{k,t_{i+1}} + \epsilon_{t_{i+1}} \]
with nonlinear restrictions on the parameters \( b_k \) (see Appendix C for details).

If \( k_c, k_p \) are estimated along with the other parameters, a nonlinear regression model has to be estimated. The resulting regression model is essentially, a nonlinear threshold model. As shown by Hansen (2000) and Linton and Seo (2006), the asymptotic distribution of the thresholds \( k_c \) and \( k_p \) is not normal. As a consequence, standard tests of significance are not applicable. In order to have a clean interpretation and not to rely on non-standard significance tests for nonlinear threshold models, it will be assumed that \( k_c = k_p = \gamma_p = 0 \) in what follows. In this case the long positions in an up-market (resp. down-market) are \( \beta + \gamma_c \) (resp. \( -\beta \)). This is because the market factor has an orthogonal decomposition in the positive and negative parts given by
\[ \log \left( \frac{1 + r_{t_{i+1}}^p}{1 + r_{t_{i+1}}^e} \right) = \left( \log \left( \frac{1 + r_{t_{i+1}}^p}{1 + r_{t_{i+1}}^e} \right) \right)^+ - \left( \log \left( \frac{1 + r_{t_{i+1}}^p}{1 + r_{t_{i+1}}^e} \right) \right)^- \]
where \( x^+ \equiv \max \{0,x\} \) and \( x^- = -\max \{-x,0\} \). Note that the same decomposition of cumulative market returns in the general model with \( \gamma_p \neq 0 \), suggests that only \( \beta + \gamma_c \) and \( -\beta + \gamma_p \), instead of \( \beta, \gamma_c \) and \( \gamma_p \), are identifiable.

HM (1981) and Breen, Jagannathan and Ofer (1986) introduce independence assumptions on the timing-ability of the active manager to obtain a regression with a single call factor and heteroscedastic conditional variance \( p (1-p) \left( \int_{t_i}^{\tau} dR_{e}^{c} \right)^2 \), where \( p \) is the probability of the timing forecast being correct. In contrast, with the method developed above which uses a continuous time embedding, the structure of the conditional volatility \( \sigma^{h,e} \) is obtained without additional assumptions. Moreover, the conditional variance of the errors in the regression model (which corresponds to \( \sigma^{h,p} \)) can be chosen constant, even if the conditional variance of hedge fund returns is time-varying.
Empirical Illustration

This section presents results for the tests of selection and timing ability and for the optimal allocation policy. The hedge fund models are estimated and tested using the database of HFRI indices available on the HFR website. The sample runs from January 1996 until August 2009. The data sets are described in Section 6.1. Results of tests of timing and selection abilities are reported in Section 6.2. Implied hedge funds characteristics and optimal portfolios are discussed in Section 6.3.

6.1 Data

A set of indices characterizing the main hedge fund strategies is considered, namely Equity Hedge, Event Driven, Macro, Relative Value, Emerging Markets and Funds of Funds. Two sub-categories are added, Equity Hedge Short Bias in the Equity Hedge family and Macro Systematic Diversified in the Macro universe. The first differs from the Equity Hedge total as it embodies strategies based on a downward market direction. The second was called Market Timing Style in the classification of HFRI indices before January 2008 and is therefore of direct relevance to the present study. The description on the HFR site states that Macro Systematic Diversified strategies “have investment processes typically as function of mathematical, algorithmic and technical models,..., designed to identify opportunities in markets exhibiting trending or momentum characteristics across individual instruments or asset classes”. Chen (2006) and Chen and Liang (2007) consider individual hedge funds of this kind and find statistically significant timing parameters in monthly TM and HM regressions. The present study uses the index as a central case for the timing tests. One would expect to find timing ability for this style.

Summary statistics for the hedge-fund indices are reported in Panel a of Table 1. For most categories, mean returns are high and standard deviations low compared to a general equity index, a typical characteristic of hedge fund returns. Exceptions are Equity Hedge, Short Bias and Emerging Markets. A small negative skewness and a strong excess kurtosis also characterize time series of hedge-fund returns. The Macro Systematic Diversified category stands out as the style with highest Sharpe ratio (1.07), together with little skewness and with lowest excess kurtosis. In comparison, Relative Value also produces a high Sharpe ratio (1.00) but is the most negatively skewed (-3.03) and has the highest excess kurtosis (18.75).

Market timing ability introduces serial correlation in returns. At the same time, as pointed out by Lo et. al (2006), certain hedge fund categories may smooth returns when they report performance, thereby introducing spurious serial correlation in their returns. In order to assess the impact of return smoothing and to separate the two sources of serial correlation, an MA(3) (moving

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14 Definitions and data can be found on https://www.hedgefundresearch.com.
15 In its reclassification of January 2008, HFR changed the label of this style, but not the data.
average) smoothing model is fitted, as in Lo et al. (2006). Unsmoothed returns are obtained by inverting the MA polynomial.\footnote{As arbitrary timing frequencies are allowed, the factor model is non-linear in parameters. The simple smoothing corrections for parameter estimates in linear factor models, proposed in Lo et al. (2006), can therefore not be applied. Instead, the model is estimated using unsmoothed data obtained from the MA(3) polynomial. The inverse polynomial, which has infinite autoregressive (AR) order, is truncated at twelve months. The AR coefficients die out sufficiently quickly so that truncation has no significant effect on the results.}

Panel b of Table 1 reports descriptive statistics for unsmoothed returns of the HFRI indices considered. The main effect of the unsmoothing procedure is to increase standard deviations and other dispersion measures, like the difference between the minimum and maximum returns. As the mean is not significantly affected, the increase in variability reduces the Sharpe ratios substantially, for categories with significant return smoothing activity. Return smoothing seems to be present in most indices, except for the two macro indices. For these two styles, the standard deviation does not increase as much and the Sharpe ratios remain high. It is also found that return smoothing does not fundamentally change the correlation structure and the rankings of Sharpe ratios, but that it accentuates the difference with unsmoothed returns for extreme realizations. This is revealed by the changes in the minimal and maximal realized return over the sample. It follows that smoothing could be an important issue for market timing and selection regressions. Further comments on this aspect are in the next section.

Summary statistics for the risk factors introduced to explain hedge-fund returns are reported in Panel b of Table 1. For the market, daily returns obtained from the Fama-French database in WRDS are used.\footnote{Wharton Research Data Service.} In accordance with Hasanhodzic and Lo (2006), other risk factors are chosen to capture peculiar risk exposures of typical hedge fund strategies. The bond factor is obtained from the return on the Barclay/Lehman U.S. Corporate AA Intermediate Bond index; the credit risk factor is calculated from the spread between the Barclay/Lehman U.S. Corporate BAA Corporate Bond index return and the Barclay/Lehman U.S. Treasury index return. A foreign exchange (FX) risk factor is also used. This factor measures the value of the US Dollar relative to a basket of foreign currencies. The commodity factor corresponds to the return on the Goldman Sachs Commodity index. In addition, the standard Fama-French size (SMB), book-to-market (HML) as well as their momentum (MOM) factors are used. Fama-French factors are obtained from the WRDS website.

### 6.2 Selection and Timing Ability

#### 6.2.1 Results for the HM Model

Hedge fund indices consist of many different hedge funds with possibly heterogeneous selection and timing abilities. The construction of equally- or value-weighted portfolios will affect the testing power of timing regressions based on index returns. If timing ability is present, it is expected to
be concentrated among hedge funds within these categories and not necessarily in a hedge fund investment category taken as a whole.

From this perspective, an empirical illustration using indices will at best be indicative of the actual timing and selection abilities of hedge funds. As discussed before, the Macro Systematic Diversified index will receive a particular focus as a potential market timing index. Moreover, it is important to note that many hedge funds attempt to time specialized indices other than the market index considered in this study (see Chen (2006) for a timing study with various focus markets). For example, styles specializing in fixed income investments most likely time a bond index rather than the equity index used in our regressions. The focus adopted here is on an equity target market alone, because data at a higher frequency than hedge fund returns are needed to implement timing tests for arbitrary timing windows and to illustrate the power of the multifrequency methodology developed. For other focus markets, daily returns are not always readily available.

Panels a and b of Table 2 display the coefficients of the timing regressions together with their t-statistics for the set of selected HFRI indices, for reported and unsmoothed returns. In the absence of controls for risk factors, it is found that the null hypothesis of timing ability cannot be rejected, for most indices, at conventional levels of confidence. The only exception is the Equity Hedge category. An interesting finding is that the only long market timing categories, with positive $\gamma$, are the two macro categories, where market timing strategies were expected to be observed. All other significant timing categories are short market timers.

Another noteworthy difference between the macro categories and the other statistically significant timing indices is the $\alpha$ coefficient, measuring selection ability. It is positive and strongly statistically significant for all indices showing short market timing ability, but not for the macro strategies. The association of a negative timing ability together with a positive selection ability has often been found with individual mutual funds (see Henriksson, 1984) or with individual hedge funds (see Diez and Garcia, 2009)). This indicates that on average, selection ability and market timing ability might be canceling each other. It therefore seems that the two macro categories, especially Macro Systematic Diversified, are representative of hedge funds that will successfully time the market with a long timing strategy.

The detection of timing or selection abilities could be a consequence of serial correlation due to return smoothing or missing risk factors. In the case of Macro, Macro Systematic Diversified and Equity Hedge Short Bias, the MA coefficients are not statistically significant, implying that return smoothing is not an issue (unreported results). Panel b of Table 2 shows that market timing ability is not statistically significant for Event Driven, Emerging Markets and Funds of Funds, if returns are unsmoothed. For Emerging Markets it cannot be rejected that selection ability is a consequence of return smoothing if the MA(3) smoothing filter is correctly specified. In fact the estimated alpha becomes negative for this category.
To further investigate the robustness of the timing results, the HM timing regression is re-estimated with the additional risk factors proposed by Lo and Hasanhodzic (2006) (bond, credit, commodity and foreign exchange) as well as the size (SMB), book-to-market (HML) and momentum (MOM) factors proposed by Fama and French (1993). Panels a and b of Table 3 report the estimation results. The timing results are seen to be robust in terms of sign, statistical significance and magnitude. Factor loadings are reported in Panel b of Table 3.

6.2.2 Results for the TM Model

Table 2, Panels c and d contain the regression estimates for the TM model, for reported and unsmoothed returns. A comparison of Panels c (reported returns) and d (unsmoothed returns) shows that results for reported and unsmoothed returns are qualitatively the same. As for HM, macro strategies are the only ones with positive market timing ability. For other categories, the signs are identical and the magnitudes roughly similar to those found in HM regressions. However, the strong statistical significance obtained for the HM model, is lost for Macro Systematic Diversified. As explained before, an HM market timer only anticipates the direction of the market whereas a TM timer foresees its actual level. It follows that TM timers are more skilled and therefore ought to be less common. This is empirically verified for the universe of hedge fund styles considered here.

6.2.3 Comparison with Regressions Based on Monthly Results

The methodology developed in this paper accounts for the possibility of timing activity at any moment during the reporting window. It uses a filter based on market returns at a higher frequency to capture and refine the structure of hedge fund returns observed at longer time intervals. To show the importance of this multifrequency approach, the previous estimates are compared to regression coefficients obtained by using monthly market returns instead of daily returns. These results are collected in Table 4. Panel a reports HM timing regressions and Panel b TM regressions.

The first striking observation is the downward bias in the magnitude of the timing coefficients when monthly returns are used. For Macro Systematic Diversified, the coefficient of $\gamma_c$ (HM) falls from 1.2711 with daily returns to 0.3333 with monthly returns. All other coefficients show similar declines in absolute values. Declines are also more pronounced with TM regressions than with HM regressions. It should however be noticed that the sign of the timing effect as well as its statistical significance are maintained when passing from one frequency to the other. The reduction in the timing coefficients is accompanied by an increase in the volatility $\sigma_{h,p}$ for both HM and TM regressions. This goes of course to the heart of the corrections needed when timing within the observation window is accounted for.
6.3 Optimal Asset Allocation

This section examines the optimal portfolio incorporating an asset class composed of hedge funds. Macro Systematic Diversified, which was found to be the most prominent category from the point of view of market timing ability, is selected as the hedge fund asset class. Optimal asset allocation with two risky asset classes, Stocks (the market) and Macro Systematic Diversified, and a riskless asset, the money market account, will hence be studied. Parameter estimates for Macro Systematic Diversified are those corresponding to raw data in Table 2. Parameters for Stocks are estimated using daily returns over the sample period, yielding $\theta^e = 0.3892$ and $\sigma^e = 0.2079$. Risk aversion is equal to $R = 5$. Dynamic hedging demands are calculated by Monte Carlo simulation using 50,000 replications and 16 time steps per day.

In order to illustrate the impact of market timing on the optimal portfolio, four stylized trajectories of cumulative market excess returns are constructed (see Figures 1-5). Each of these trajectories has a length of 22 days, the length of the timing window. Dependent on the path of cumulative returns, the scenarios are labeled “up-up”, “down-down”, “up-down” and “down-up”.\(^{18}\) The up-up (resp. down-down) scenario corresponds to a persistent bull (resp. bear) market. The other two scenarios are examples of mixed market situations. The mixed market scenarios, “up-down” and “down-up”, are constructed so that cumulative excess market returns after 14 days are null. This helps to highlight the path-dependent nature of the allocation policy.

6.3.1 Correlation and Static Hedging Demand

Figure 1 shows the evolution over the timing window of the instantaneous correlation coefficient between hedge fund and market returns in the 4 scenarios, for the HM (left panel) and TM (right panel) timing models. The behavior across models is strikingly different. In the HM model, the correlation exhibits jumps and grows deterministically in between jump times. In TM, the behavior is more regular and parallels the evolution of cumulative returns. The explanation for these diverging patterns is directly related to the skill of the hedge fund manager assumed in the two models. In HM, the manager is endowed with the directional ability to separate up and down markets. The resulting correlation coefficient depends on $\sqrt{v - t} 1_{R^e_{t,v} > 0}$, where $R^e_{t,v} \equiv \int_t^v dR^e_v$. It increases as $\sqrt{v - t}$ as long as the cumulative return is positive.\(^{19}\) At times at which the cumulative return turns negative, the correlation drops to its baseline value related to the beta of the fund. In TM, the manager has the additional ability to time the level of the market. The implied correlation coefficient depends on the cumulative return $R^e_{t,v}$.

The mean-variance market weight moves in the opposite direction of the correlation coefficient.

\(^{18}\)The optimal portfolio is path-dependent (infinite dimensional). The paths selected here are examples used to illustrate some of its properties.

\(^{19}\)Recall that scaling is necessary to ensure the absence of arbitrage opportunities.
This is due to the static hedging component in the mean-variance demand, which is determined by the comovement between market and hedge fund returns and by the market price of hedge fund risk. In the TM model, the market price of hedge fund risk is a constant. It follows that the evolution of the static hedging component is fully determined by the dynamics of the conditional correlation coefficient. The resulting evolution of the market weight is the mirror image of the evolution of the correlation coefficient scaled by relative risk aversion. In a persistent bull market, the market correlation induced by a skilled timer increases over time, thereby raising the exposure of a hedge fund position to market risk. Optimal hedging requires a reduction in the position taken in stocks. Similar patterns and explanations apply in other market scenarios.

The dynamic portfolio patterns observed in the HM model are more involved. They reflect the behavior of the correlation coefficient as well as the behavior of the market price of hedge fund risk. In the absence of a market price of risk effect the static hedging component behaves as in the TM model. It offsets the exposure to market risk associated with the position in the hedge fund. The dependence on the market price of hedge fund risk introduces an additional source of variations. As discussed previously, the behavior of the market price of risk reflects a Delta effect and a Theta of Delta effect, with opposite signs. The Delta effect dominates early during the timing window, implying a market price of hedge fund risk that follows the cumulative return pattern. During this part of the window, the static hedging component behaves as in the TM model. Towards the end of the timing window, the non-monotonic Theta of Delta effect dominates, causing more pronounced fluctuations in the endogenous market price of risk. This effect is most pronounced in the persistent bull market scenario. In that scenario it results in substantial variations in the static hedging component, that are opposite to those in cumulative returns.

### 6.3.2 Hedge Fund Volatility and Market Price of Hedge Fund Risk

Figures 2-5 focus on the HM timer and show more detailed results for the 4 scenarios of interest.

As discussed previously, the hedge fund volatility coefficient $\sigma_{h,e}$ is an affine function of $\sqrt{v - T}$ as long as the cumulative return is positive. When the cumulative return is negative volatility remains constant at the base level $\beta \sigma^e$. This pattern can be observed in the second rows in Figures 2-5 (left panels). The behavior parallels the behavior of the correlation coefficient already documented in Figure 1.

It may be useful to emphasize that the behavior of the market volatility of the hedge fund return influences the behavior of the covariation between market and hedge fund returns. The presence of a time-varying component in correlation is an important feature missed by linear factor models assuming constant coefficients. If cumulative returns are positive, a hedge fund timing the market will be long in the market. This position will increase both the volatility of the returns and their correlation with the market. Recent empirical papers by Bilio et al. (2006) and Boyle et
al. (2006) propose a Markov-switching regression model to analyze hedge fund returns. In these models, the volatility is exogenously specified at different levels in each regime. In contrast, the analysis carried out here explains why different regimes may be apparent in the data and shows the relations between the components of volatility and correlation.

The market price of idiosyncratic hedge fund risk $\theta^h$ is increasing in cumulative returns for dates close to the start of the timing window. This is the behavior observed in the second rows in Figures 2-5 (right panels). However, as shown by the “down-up” scenario in Figure 4, $\theta^h$ can be decreasing, even if cumulative returns are positive and increasing. This illustrates the dominance of the Theta of Delta effect over the Delta effect close to the end of the timing window. The dependence on realized market returns can also be the source of important fluctuations in the market price of risk. This is especially apparent in the persistent bull market scenario in Figure 2.

### 6.3.3 Optimal Portfolios with Timing Ability

The last two rows of Figures 2-5 show the optimal portfolio weights, unconstrained (third rows) and short sales constrained (fourth rows). Left panels display the market weights (bold line) and mean-variance weights (dotted line), right panels show the hedge fund weights.

As indicated above, investments in the hedge fund are entirely motivated by mean-variance considerations and therefore driven by the market price of hedge fund risk. In all scenarios the demand for Macro Systematic Diversified mimics the behavior of this market price of risk. It is increasing in cumulative returns, except if the investment date is far away from the beginning of the timing window. Close to the beginning of the window, it is optimal to hold the hedge fund short (resp. long), when cumulative returns are negative (resp. positive). If the initial investment date is further back, additional maturity effects due to Theta of Delta can become dominant and render short sales optimal even when cumulative returns are positive and increasing. Furthermore, the non-monotonicity appears even in persistent bull or bear markets. This behavior simply reflects the fact that the cumulative return is not a sufficient statistic for the optimal allocation to funds with significant market timing ability.

The demand for Stocks reflects mean-variance as well as dynamic hedging considerations. The importance of the dynamic hedging motive can be seen in the left panels of the figures (third rows). The dynamic hedge is the difference between the total demand for Stocks (bold line) and the mean-variance component (dotted line). In the early part of the timing window, when the Delta effect dominates, the dynamic hedge is negatively related to the cumulative market return. It is negative (positive) in bull (bear) markets, thus decreases (increases) the demand for Stocks. At later times, the Theta of Delta effect can become more pronounced and invert the hedging pressure. This is especially apparent in the persistent bull and bear markets depicted in Figures 2 and 3. Even though the market keeps increasing (Figure 2) or decreasing (Figure 3) the hedge eventually
increases or reduces the demand for Stocks relative to the mean-variance component. In the mixed market scenarios of Figures 4 and 5 the Delta effect remains prominent even late during the timing window. In these scenarios cumulative returns remain small at later times (due to the inversion) implying smaller Theta of Delta hedging components. Eventually all dynamic hedging vanishes: as the end of the window approaches the demand for Stocks is entirely motivated by mean-variance considerations.

The impossibility to short sell Macro Systematic Diversified has a significant effect on the dynamic hedging demand (see last rows of Figures 2-5). In the presence of the no short sales constraint, the demand for Stocks has two dynamic hedging components. The first is motivated by stochastic fluctuations in the market price of hedge fund risk $\theta^h$. The second is designed to hedge fluctuations in the shadow price of the no-short sales constraint $\theta^h - (\theta^h)^+ = - (\theta^h)^-$, which corresponds to the negative part of the market price of hedge fund risk. This constraint-induced hedge is always positive at the beginning of the timing window (compare the third and fourth rows in all the figures). It is also significant in size. In some cases it offsets the negative dynamic hedge associated with fluctuations in the market price of risk. This is the pattern observed during the bull market stretches in Figures 2 and 5, where this shadow price hedge increases the demand by up to 12%. Towards the end of the timing window it reduces the magnitude of the combined dynamic hedge. As the end of the window approaches, the shadow price hedge vanishes.

The broad conclusion emerging from this analysis is that timing ability has a significant impact on the portfolio composition. Salient aspects include the following. First, preconized Hedge fund and Stock investments are fundamentally different from those suggested by a classic mean-variance analysis based on unconditional Sharpe ratios. Second, hedging motives are found to be essential for sound asset allocation. They drive part of the mean-variance component of the Stock demand, as well as its dynamic hedging component. The hedging behavior is entirely triggered by the endogenous timing activity/skill present in Macro Systematic Diversified and is different in nature from that induced by stochastic fluctuations in asset return coefficients.

6.3.4 Optimal Portfolios with Credit and HML Factors

Credit and HML were found to be the most relevant risk factors for Macro Systematic Diversified. The parameter estimates, reported in Figure 6 are all statistically significant. Factor loadings are negative. This suggests that the category is short defaultable corporate bonds and growth stocks and long Treasuries and value stocks. The exposure to credit risk is more pronounced.

The optimal portfolio composition and behavior are illustrated in Figure 6. Only constrained

\[20\text{Standard mean-variance analysis suggests highly levered portfolio with } -41.61\% \text{ of wealth in Stocks and } 313.34\% \text{ in Macro Systematic Diversified.}
\]

\[21\text{The credit factor does not separate liquidity risk from credit risk. The factor loading could therefore indirectly measure the exposure to the recent increase in liquidity risk.}
\]
portfolios are graphed (no short sales of hedge fund category). The left column shows all the components, Stocks, Macro Systematic Diversified, Credit and HML, when factors are traded. The middle column focuses on the demand for Stocks (total and mean-variance component) when factors are traded. The right column gives the demand for Stocks (total and mean-variance) when factors are non-traded. The rows correspond to the 4 scenarios.

Investments in the factors have a mean-variance structure, which incorporates static hedges. The static hedges are due to the demand for Macro Systematic Diversified which, if positive, creates a negative exposure to Credit and HML. When short sales of the hedge fund category are precluded, the exposure to these factors is always negative. Static hedges offset this negative exposure and are therefore positively related to the demand for the hedge fund. This positive association is clear in Figure 7. The investment in the Credit factor is nevertheless negative, static hedges being offset by other negative terms in the mean-variance components. The investment in the HML factor is also negative, but only during part of the timing window. In order to explain these patterns in more detail, note that the Sharpe ratios for both Credit and HML are negative ($\theta_{\text{Credit}} = -0.6886$ and $\theta_{\text{HML}} = -0.0247$) and that the HML factor is negatively correlated with the market portfolio ($\rho_{\text{Credit,HML}} = -0.0094$). The associated demand components are negative. In contrast, static hedges are positive. This is because factor loadings and the market price of hedge fund risk are both negative. In terms of size, the third row of the figure shows that the static hedge for Credit (resp. HML) is at most 20% (resp. 8%). The negative terms dominate for Credit. The positive static hedge is the dominant effect for HML around the peak of the boom period.

Credit and HML have a marginal effect on the point estimates of parameters in the HM model. As these parameters determine the market price of hedge fund risk, optimal investments in Macro Systematic Diversified are not significantly altered. Comparison of the middle panels in Figure 6 to the left panel in the last row of Figures 2-5 shows that the demand for Stocks increases slightly in all scenarios. This is due to the static hedges associated with the exposure to Credit and HML in the demand for Stocks.

The right panels of Figure 6 show the demand for Stocks and its mean-variance component when factors are nontraded. Their difference is the dynamic hedging component. As the covariances between the market and the factors are constant, the evolution of the dynamic hedge is the same as in the case of traded factors. But the magnitude is considerably different. In fact, nontradedness boosts both components of the demand for Stocks. The extent of the combined effect is best seen by comparing the left panels in rows 2 and 4 of Figure 7.

The addition of Credit and HML factors has no qualitative effect on the evolution of the demands for Stocks and Macro Systematic Diversified. From a quantitative point of view, nontradedness boosts the mean-variance demand for Stocks. Credit risk, in particular, seems to have a large impact. This may be specific to the sample period considered, which includes the recent credit
crunch. A more refined, conditional factor structure could be introduced to capture extreme events of this sort. Extending the model in that direction would introduce additional hedging demands, distinct from those related to the timing skill of the hedge fund.

6.4 The Economic Value of Hedge Funds

To complete this study, the economic relevance of hedge funds for outside investors is assessed. The analysis of the previous sections suggests that an asset class based on hedge funds can play an important role for the optimal allocation of risks. The next proposition compares the (wealth) certainty equivalents for an investor with access to hedge funds relative to one without.

Proposition 10. Let \( \hat{x}^h \) (resp. \( \hat{x}^e \)) be the certainty equivalent for an investor with constant relative risk aversion \( R \), initial wealth \( x \) and investment horizon \( T \), who invests (resp. does not invest) in hedge funds. The ratio of certainty-equivalents is

\[
\frac{\hat{x}^h}{\hat{x}^e} = \begin{cases} 
    E \left[ \exp \left( -\rho \theta^e W_T - \frac{1}{2} (\rho \theta^e)^2 T + \frac{1}{2} \rho (\rho - 1) \int_0^T \left( \theta_{s,c}^h \right)^2 ds \right) \right]^{-1/\rho} & \text{HM strategies} \\
    \exp \left( -\frac{1}{2} (\rho - 1) \int_0^T \left( \theta_{s,c}^h \right)^2 ds \right) = \exp \left( \frac{1}{2R} \left( \frac{\alpha \sigma_{h,p} \gamma^{TM} (\sigma^e)^2}{\sigma_{h,p}} \right)^2 T \right) & \text{TM strategies}
\end{cases}
\]

where \( \rho = 1 - 1/R \) and \( \theta_{s,c}^h \) is the (constrained) market price of hedge fund risk.

The ratio of certainty equivalents is increasing in the total performance measure (market price of hedge fund risk) and decreasing in relative risk aversion. Thus, funds with greater timing ability \( \gamma \) and positive idiosyncratic volatility \( \sigma_{h,p} \) are of greater value to outside investors.

The magnitude of the economic gains realized is presented in Figure 8 that shows the economic value of the Macro Systematic Diversified style. The left Panel in Figure 8 shows significant gains from investing in hedge funds, even for very short horizons. If market-timers operate with a monthly timing window, an outside investor with risk aversion \( R = 2 \) (resp. \( R = 4 \)) with an investment horizon of 1 year, will increase his/her certainty equivalent by 11.52% (resp. 5.60%). With a horizon of 2 years these economic gains reach 24.37% (resp. 11.51%). With a horizon of 3 years they soar to 38.71% (resp. 17.25%).

The right Panel in Figure 8 quantifies the relative cost of the short-sale constraint, expressed as the reduction in the ratios of certainty equivalents implied by the constraint. For \( R = 2 \) (resp. \( R = 4 \)) constrained and unconstrained certainty equivalents differ by 7.42% (resp.3.46%) for an investment horizon of 1 year, by 17.11% (resp. 7.42%) for 2 years, and by 29.58% (resp. 11.95%) for three years. The costs of the short sale constraint are clearly considerable.
7 Conclusion

This paper has examined the impact of market timing factors in hedge fund returns on optimal asset allocation policies. The presence of market timing was shown to imply path-dependence in the market price of hedge fund risk. When market returns have independent increments the market price of hedge fund risk in the TM timing model reduces to the Treynor-Mazuy total performance measure, which is constant. In the HM timing model it remains non-Markovian and time-varying. The resulting optimal portfolio contains a novel hedging component, motivated by the market timing behavior of the hedge fund manager. The structure of the optimal portfolio is derived under a variety of regulatory and economic conditions, including the presence of no short sales constraints and the presence of nontraded factors in hedge fund returns.

Portfolio results are illustrated using bias corrected and unsmoothed hedge fund data. The optimal fraction of wealth in the stock market is shown to deviate considerably from that predicted by standard mean-variance analysis à la Markovitz. Deviations were found to vary widely depending on business conditions (bull versus bear markets) and the time elapsed since the start of the timing window. The cost of ignoring hedge funds altogether was shown to be significant. The cost of incorporating hedge funds but pursuing a simple minded mean-variance allocation strategy can also be shown to be significant. Overall, the study suggests that carefully selected hedge funds can be used to constitute a useful asset class that provides significant benefits to investors and should be incorporated in asset allocation strategies of institutional investors and pension plans.

This study has important practical implications for institutional investors and pension plan managers seeking to enhance performance by investing in hedge fund asset classes. First and foremost, it shows that investing in selected hedge fund categories can be the source of substantial economic benefits. Second, it also underscores the fact that maintaining and managing separate portfolios in traditional asset classes and alternative investments is suboptimal. Investments in hedge funds modify the exposure to market risk and other factors, mandating a readjustment of traditional portfolio allocations. Finally, it shows that hedge fund strategies of successful market timers require monitoring and rebalancing of asset positions. Even a myopic investor will find it necessary to rebalance his/her mean-variance allocations in order to extract all the benefits provided by access to hedge fund returns.
Appendix A: Timing with stochastic opportunity set

Previous sections consider the TM and HM model in a Black-Scholes setting where market returns follow a geometric Brownian motion. This appendix generalizes the results to arbitrary market dynamics with stochastic coefficients \((r_t, \theta^e_t, \sigma^e_t)\). The next proposition gives the implied hedge fund parameters for TM and HM strategies in this context.

**Proposition 11.** Consider a financial market with stochastic coefficients \((r_t, \theta^e_t, \sigma^e_t)\). Hedge fund volatilities and market prices of risk are given by

\[
\sigma^e_{t,v} = \kappa_{t,v} \sigma^e_v \quad \text{with} \quad \kappa_{t,v} \equiv \beta_t + \gamma_t \Delta_v(t, v)
\]

\[
\theta^h_{t,v} = \theta^h_{t,v}^S + \theta^h_{t,v}^T = \frac{\alpha_{t,v}}{\sigma^h_{t,v}} + \gamma_t \left( \frac{\Theta_t(t, v) + \int_t^v \partial_s(s, v) dR^e_s}{\sigma^h_{t,v}} \right)
\]

where \(\Delta_v(t, v), \Theta_t(t, v), \partial_s(s, v)\) are

\[
\Delta_v^{TM}(t, v) = 2R^e_{t,v}
\]

\[
\Theta^{TM}_t(t, v) = \frac{\partial C^{TM}_t(t, v)}{\partial v} = E_t \left[ Z_{t,v}(\sigma^e_v)^2 \right]
\]

\[
\partial^{TM}_t(t, v) = E_t \left[ Z_{t,v} \left( 2\sigma^e_v \partial^e_v + (\sigma^e_v)^2 \int_t^v dR^e_s (\sigma^e_s)^{-1} \partial^e_s \right) \right]
\]

for Treynor-Mazuy strategies and

\[
\Delta^c_s(t, v, k_{ct}) = (\sigma^c_s)^{-1} \partial^c_s C^{HM}_s(t, \tau, k_{ct}), \quad \Delta^p_s(t, v, k_{pt}) = (\sigma^c_s)^{-1} \partial^p_s P^{HM}_s(t, \tau, k_{pt})
\]

\[
\Theta^c_s(t, v, k_c) = \frac{\partial C^{HM}_s(t, v, k_c)}{\partial v}, \quad \Theta^p_s(t, v, k_p) = \frac{\partial P^{HM}_s(t, v, k_p)}{\partial v}
\]

\[
\partial^c_s(t, v, k_c) = \frac{\partial \Theta^c_s(t, v, k_c)}{\partial v}, \quad \partial^p_s(t, v, k_p) = \frac{\partial \Theta^p_s(t, v, k_p)}{\partial v}
\]

\[
C^{HM}_s(t, \tau, k_{ct}) \equiv E_s \left[ Z_{s,\tau} \left( R^e_{s,\tau} - k_{ct}(\tau - t) \right)^+ \right]
\]

\[
P^{HM}_s(t, \tau, k_{pt}) \equiv E_s \left[ Z_{s,\tau} \left( k_{pt}(\tau - t) - R^e_{s,\tau} \right)^+ \right]
\]

for Henriksson-Merton strategies. In these expression \(\partial^e_t \sigma^e_v, \partial^e_t \theta^e_v\) are the Malliavin derivatives of the market coefficients \((\sigma^e_v, \theta^e_v)\).

With stochastic opportunity sets the general structure of hedge fund volatilities and market prices of risk remains the same. The ingredients entering these formulas are nevertheless affected. The most apparent effect is in the TM model, in which \(\partial^{TM}_t\) is no longer null. This reflects the stochastic behavior of \(\sigma^e_v\) and \(\theta^e_v\). The resulting performance measure and the market price of hedge fund risk become path-dependent. In the HM model the structure of the Greeks changes as well, reflecting the behavior of \((\sigma^e_v, \theta^e_v)\).
Appendix B: Market Price of Risk in Henriksson-Merton

For any \( u \in [t, v] \), the market price of risk has the decomposition \( \theta_{t,u}^{HM} = \theta_{1,t,u,v}^h + \theta_{2,t,u,v}^h \) where

\[
\theta_{1,t,u,v}^h = \mathbb{E} \left[ \frac{Z_v}{Z_u} \theta_{t,v} \bigg| \mathcal{F}_u^c \right] \quad \text{and} \quad \theta_{2,t,u,v}^h = \theta_{t,v}^h - \mathbb{E} \left[ \theta_{t,v}^h \bigg| \mathcal{F}_u^c \right].
\]

The term \( \theta_{1,t,u,v}^h \) is an \( \mathcal{F}_u^c \)-measurable projection; \( \theta_{2,t,u,v}^h \) is an innovation. Defining \( d(k, v - u) \), \( e(R; k, u, v, s) \) and \( l(R; k, u, v, s) \) as in (31)-(32), enables us to write

\[
\theta_{1,t,u,v}^h = \frac{\alpha}{\sigma_{h,p}} - \frac{3}{2} \frac{\sigma^e}{\sigma_{h,p}} (\gamma_c d(k_c, v - t) \Phi(-d(k_c, v - t)) - \gamma_p d(k_p, v - t) \Phi(d(k_p, v - t))) \\
+ \frac{1}{2\sigma_{h,p}} \int_t^u \left( \gamma_c \Phi(-e(R_{t,s}; k_c, t, u, v, s)) - \gamma_p \Phi(e(R_{t,s}; k_p, t, v, s)) \right) dR_s^e \\
+ \frac{\gamma_c \sqrt{v - t}}{\sigma_{h,p}} \int_t^u e(R_{t,s}; k_c, t, v, s) l(R_{t,s}; k_c, t, v, u) dR_s^e \\
+ \frac{\gamma_p \sqrt{v - t}}{\sigma_{h,p}} \int_t^u e(R_{t,s}; k_p, t, v, s) l(R_{t,s}; k_p, t, v, u) dR_s^e
\]  

(72)

\[
\theta_{2,t,u,v}^h = \frac{1}{2\sigma_{h,p}} \int_u^v \left( \gamma_c \Phi(-e(R_{t,s}; k_c, t, v, s)) - \gamma_p \Phi(e(R_{t,s}; k_p, t, v, s)) \right) dR_s^e \\
+ \frac{\gamma_c \sqrt{v - t}}{\sigma_{h,p}} \int_u^v e(R_{t,s}; k_c, t, v, s) l(R_{t,s}; k_c, t, v, u) dR_s^e \\
+ \frac{\gamma_p \sqrt{v - t}}{\sigma_{h,p}} \int_u^v e(R_{t,s}; k_p, t, v, s) l(R_{t,s}; k_p, t, v, u) dR_s^e.
\]  

(73)

Malliavin derivatives can then be calculated as \( D_u^e \theta_{t,u,v} = D_u^e \theta_{1,t,u,v}^h + D_u^e \theta_{2,t,u,v}^h \) with

\[
D_u^e \theta_{1,t,u,v}^h = \frac{\sigma^e}{2\sigma_{h,p}} \int_t^u \left( \gamma_c \Phi(-e(R_{t,s}; k_c, t, v, s)) - \gamma_p \Phi(e(R_{t,s}; k_p, t, v, s)) \right) dR_s^e \\
+ \frac{\gamma_c \sqrt{v - t}}{\sigma_{h,p}} \phi(e(R_{t,s}; k_p, t, v, s)) l(R_{t,u}; k_p, t, v, u) \\
+ \frac{\gamma_p \sqrt{v - t}}{\sigma_{h,p}} \phi(e(R_{t,s}; k_p, t, v, s)) l(R_{t,u}; k_p, t, v, u)
\]  

(74)
\[
D^e_{u} \theta^h_{2,t,u,v} = \gamma_c \frac{1}{2 \sigma^h \nu} \int_u^v \frac{1}{\sqrt{v-s}} \phi \left( e(R^e_{t_i}; k_c, t, v, s) \right) dR^e_s \\
+ \gamma_p \frac{1}{2 \sigma^h \nu} \int_u^v \frac{1}{\sqrt{v-s}} \phi \left( e(R^e_{t_i}; k_p, t, v, s) \right) dR^e_s \\
- \gamma_c \frac{\sqrt{v-t}}{\sigma^h \nu} \int_u^v \phi \left( e(R^e_{t_i}; k_c, t, v, s) \right) l(R^e_{t_i}; k_c, t, v, s) dR^e_s \\
- \gamma_p \frac{\sqrt{v-t}}{\sigma^h \nu} \int_u^v \phi \left( e(R^e_{t_i}; k_p, t, v, s) \right) l(R^e_{t_i}; k_p, t, v, s) dR^e_s \\
+ \gamma_c \frac{\sqrt{v-t}}{\sigma^h \nu} \int_u^v \phi \left( e(R^e_{t_i}; k_c, t, v, s) \right) l(\sigma^e; k_c, t, v, s) dR^e_s \\
+ \gamma_p \frac{\sqrt{v-t}}{\sigma^h \nu} \int_u^v \phi \left( e(R^e_{t_i}; k_p, t, v, s) \right) l(\sigma^e; k_p, t, v, s) dR^e_s. \\
\]

(75)

Appendix C: Implementation of Timing Models

C.1 Treynor-Mazuy Timing Strategies

Using (67) and (69) gives

\[
\log \left( \frac{1 + r^h_{t_i+1}}{1 + r^f_{t_i+1}} \right) = \left( \alpha - \frac{1}{2} (\sigma^h)^2 + \frac{\beta}{2} (1 - \beta) (\sigma^e)^2 \right) \Delta - \left( \frac{1}{2} \beta \gamma^T M (\sigma^e)^4 - \frac{\gamma^T M}{4} (\sigma^e)^4 \right) \Delta^2 \\
- \frac{1}{6} (\gamma^T M)^2 (\sigma^e)^6 \Delta^3 - 2 \beta \gamma^T M (\sigma^e)^2 \int_{t_i}^{t_{i+1}} \log \left( \frac{1 + r^e_v}{1 + r^f_v} \right) dv \\
- 2 (\gamma^T M)^2 (\sigma^e)^4 \int_{t_i}^{t_{i+1}} (v - t_i) \log \left( \frac{1 + r^e_v}{1 + r^f_v} \right) dv \\
- 2 (\gamma^T M)^2 (\sigma^e)^2 \int_{t_i}^{t_{i+1}} \log \left( \frac{1 + r^e_v}{1 + r^f_v} \right)^2 dv \\
+ (\beta + \gamma^T M (\sigma^e)^2 \Delta) \log \left( \frac{1 + r^f_{t_i+1}}{1 + r^f_{t_i+1}} \right) + \gamma^T M \left( \log \left( \frac{1 + r^f_{t_i+1}}{1 + r^f_{t_i+1}} \right) \right)^2.
\]

This expression, along with the definitions

\[
b_0 \equiv \left( \alpha - \frac{1}{2} (\sigma^h)^2 + \frac{\beta}{2} (1 - \beta) (\sigma^e)^2 \right) \Delta \\
- \left( \frac{1}{2} \beta \gamma^T M (\sigma^e)^4 - \frac{\gamma^T M}{4} (\sigma^e)^4 \right) \Delta^2 - \frac{1}{6} (\gamma^T M)^2 (\sigma^e)^6 \Delta^3 \\
b_1 \equiv -2 \beta \gamma^T M (\sigma^e)^2 \\
b_2 \equiv -2 (\gamma^T M)^2 (\sigma^e)^4 \\
b_3 \equiv -2 (\gamma^T M)^2 (\sigma^e)^2 \\
b_4 \equiv (\beta + \gamma^T M (\sigma^e)^2 \Delta) \\
b_5 \equiv \gamma^T M
\]

42
C.2 Henriksson-Merton Timing Strategies

Using (70) and (71), gives

\[ S_{t+1}^h = \left( \alpha - \frac{1}{2} \left( \sigma^h p \right)^2 + \beta \left( \sigma - 1 \right) \sigma^e \right) \Delta + \beta \log \left( \frac{1 + r_{t+1}^e}{1 + r_t^e} \right) + \sigma^h p (\tilde{W}_{t+1} - \tilde{W}_t) \]

This, combined with the definitions

\[ b_0 \equiv (\alpha - \frac{1}{2} \left( \sigma^h p \right)^2 + \beta (\sigma - 1) \sigma^e) \Delta \quad b_1 \equiv \beta \quad b_2 \equiv -\beta \sigma^e \gamma_c \]
\[ b_3 \equiv -\beta (\sigma^e)^2 \gamma_p \quad b_4 \equiv -(\sigma^e)^2 \gamma_p \gamma_c \quad b_5 \equiv -\sigma^e \gamma_c \]
\[ b_6 \equiv \frac{(\sigma^e)^2 \gamma_p^2}{2} \quad b_7 \equiv \gamma_c \sqrt{\Delta} \quad b_8 \equiv \gamma_p \sqrt{\Delta} \]
\[ Y_{t_{i+1}} \equiv \log \left( \frac{1+r_{t_{i+1}}^e}{1+r_{t_{i+1}}^c} \right) \]
\[ X_{1,t_{i+1}} \equiv \log \left( \frac{1+r_{t_{i+1}}^e}{1+r_{t_{i+1}}^c} \right) \]
\[ X_{3,t_{i+1}} \equiv \int_{t_i}^{t_{i+1}} \sqrt{v-t_i} 1_{\{R_{t_i,v}^e<k_p(v-t_i)\}} \, dv \]
\[ X_{5,t_{i+1}} \equiv \int_{t_i}^{t_{i+1}} (v-t_i) 1_{\{R_{t_i,v}^e<k_p(v-t_i)\}} \, dv \]
\[ X_{7,t_{i+1}} \equiv \left( \log \left( \frac{1+r_{t_{i+1}}^e}{1+r_{t_{i+1}}^c} \right) - (k_c - \frac{1}{2}(\sigma^e)^2) \Delta \right) + \]
\[ X_{8,t_{i+1}} \equiv \left( (k_p - \frac{1}{2}(\sigma^e)^2) \Delta - \log \left( \frac{1+r_{t_{i+1}}^e}{1+r_{t_{i+1}}^c} \right) \right) + \]
\[ \epsilon_{t_{i+1}} = \sigma^{h,p}(\tilde{W}_{t_{i+1}}^p - \tilde{W}_t^p) \]

leads to the linear regression model (76) for log-price changes with parameter restrictions

\[ b_0 \equiv \alpha - \frac{1}{2}((\sigma^{h,p})^2 + b_7^2(\sigma^e)^2) \]
\[ b_1 \equiv -(\sigma^e)^2b_7b_8 \]
\[ b_2 \equiv -b_1(\sigma^e)^2b_7 \]
\[ b_3 \equiv -b_1(\sigma^e)^2b_8 \]
\[ b_5 \equiv -\frac{(\sigma^e)^2b_7^2}{2\Delta} \]
\[ b_6 \equiv -\frac{(\sigma^e)^2b_8^2}{2\Delta} \]

**Appendix D: Proofs**

**Proof of Proposition 1:** Let \( C_s(t,\tau) \equiv \mathbb{E}_s[Z_{s,\tau}f(t, R_{t,\tau}^e)] \). The Clark-Ocone formula (see Nualart (1995)) shows that

\[
C_s(t,\tau) = C_t(t,\tau) + \int_t^s \mathbb{E}_v[D_v^eC_s(t,\tau)] \, dW_v^e
\]

\[
= C_t(t,\tau) + \int_t^s D_v^e(Z_{t,v}\mathbb{E}_v[Z_{v,\tau}f(t, R_{t,\tau}^e)]) \, dW_v^e
\]

\[
= C_t(t,\tau) + \int_t^s Z_{t,v} (D_v^eC_v(t,\tau) - C_v(t,\tau)(\theta_v^e)^\prime) \, dW_v^e.
\]

where \( D_v^e \) is the Malliavin derivative operator with respect to \( W^e \). The second equality follows from the commutativity of the conditional expectation and Malliavin derivative operators. Applying Ito’s lemma to \( Z_{t,\tau}^{-1}C_v(t,\tau) \equiv f(t, Z_{t,\tau} R_{t,\tau}^e) \), then gives

\[
f(t, R_{t,\tau}^e) = C_t(t,\tau) + \int_t^s D_v^eC_v(t,\tau) (dW_v^e + \theta_v^e \, dv)
\]

for \( \tau \in [t,T] \). This, together with \( dW_v^e + \theta_v^e \, dv = (\sigma_v^e)^{-1} \, dR_v^e \), \( \Theta_t(t,v) \equiv \partial C_v(t,v)/\partial v \) and \( \partial D_v^eC_t(t,v)/\partial v = D_v^e\Theta_t(t,v) \), shows

\[
df(t, R_{t,v}^e) = \left( \Theta_t(t,v) + \int_t^v D_s^e\Theta_s(t,v) (\sigma_v^e)^{-1} \, dR_s^e \right) \, dv + D_v^eC_v(t,v)(\sigma_v^e)^{-1} \, dR_v^e.
\]

Substituting this expression in (9), identifies the volatility coefficient \( \sigma_{t,v}^{h,e} \). Similarly, given that the drift of the hedge fund return process is \( \sigma_v^{h,p}\theta_v^h \), the regression (9) identifies the market price of hedge fund risk. The expression for \( \sigma_{t,v}^{h,e} \) leads to the correlation coefficient (19).
Proof of Proposition 11: In the TM model \( f(t, R^e_{t,v}) = (R^e_{t,v})^2 \) and the timing option value is

\[
C^TM_v(t, \tau) = E_v \left[ Z_{v,\tau} \left( \int_t^\tau dR^e_s \right)^2 \right] = (R^e_{t,v})^2 + C^TM_v(v, \tau).
\]

Given that \( C^TM_v(v, v) = 0 \), it then follows that \( D^e_v C^TM_v(t, v) = 2R^e_{t,v} \sigma^e_v \) and

\[
\kappa^TM_{t,v} = \beta^TM_t + 2\gamma^TM_v R^e_{t,v}.
\]

Similarly, \( C^TM_t(t, v) = \int_t^\tau \mathbf{E}_t^Q \left[ (\sigma^e_v)^2 \right] ds \), leads to

\[
\Theta^TM_t(t, v) = \frac{\partial C^TM_t(t, v)}{\partial v} = \mathbf{E}_t^Q \left[ (\sigma^e_v)^2 \right] = \mathbf{E}_t \left[ Z_{t,v} (\sigma^e_v)^2 \right]
\]

\[
\psi^TM_t(t, v) \equiv D_t^e \Theta^TM_t(t, v) = \mathbf{E}_t \left[ Z_{t,v} \frac{2\sigma^e_v (D^e_t \sigma^e_v) + (\sigma^e_v)^2}{1} \int_t^v (dW^e_s + \theta^e_s ds) D^e_t \theta^e_s \right].
\]

Consider now the HM model. The expression for \( \kappa^HM_{t,v} \) follows from \( D^e_v C^V_v(t, v) = D^e_v f(t, R^e_{t,v}) = f' (R^e_{t,v}) \sigma^e_v \) where \( f' (x) = \gamma_c 1_{x > \kappa_c (v-t)} - \gamma_p 1_{x < \kappa_p (v-t)} \). The formula for \( \theta^HM_{t,v} \) is obtained from the expression in Proposition (1) with two extra terms, \( \frac{1}{2\sqrt{v-t}} \int_{\lambda (v-t)}^v (\sqrt{v-t} C_t(v, k_c) + \gamma_p P_t (t, v, k_p)) \) and

\[
\frac{1}{2\sqrt{v-t}} \int_{\lambda (v-t)}^v (\sqrt{v-t} D^e_t C_t(v, k_c) + \gamma_p D^e_t P_t (t, v, k_p) (\sigma^e_v)^{-1}) dR^e_s
\]

due to the scaling of the timing-payoff function \( f \) by \( v - t \).

Proof of Corollary 2: When \( \sigma^e_v = \sigma^e \) is constant, \( \Theta_t (t, v) = \mathbf{E}_t^Q [ (\sigma^e_v)^2 ] = (\sigma^e)^2 \) and \( \psi_t (t, v) = 0 \).

Proof of Corollary 3: Note that \( C_s (t, v) = \mathbf{E}_s^Q \left[ (\int_s^v dR^e_u - K_s (v-t))^+ \right] \), where \( K_s \equiv k_c - R^e_{t,s} (v-t) \), and that the law of \( \int_s^v dR^e_u \) in the measure \( \mathbf{Q}_s \) is \( \mathcal{N} \left( 0, (\sigma^e)^2 (v-s) \right) \). Hence,

\[
C_s (t, \tau) = \int_{K_s (v-t)}^{+\infty} \frac{x}{\sigma^e \sqrt{v-s}} \phi \left( \frac{x}{\sigma^e \sqrt{v-s}} \right) dx
\]

\[
- K_s (v-t) \int_{K_s (v-t)}^{+\infty} \phi \left( \frac{x}{\sigma^e \sqrt{v-s}} \right) d \left( \frac{x}{\sigma^e \sqrt{v-s}} \right)
\]

\[
= \sigma^e \sqrt{v-s} \int_{K_s (v-t)/(\sigma^e \sqrt{v-s})}^{+\infty} x \phi (x) dx - K_s (v-t) \int_{K_s (v-t)/(\sigma^e \sqrt{v-s})}^{+\infty} \phi (x) dx
\]

\[
= \sigma^e \sqrt{v-s} \int_{K_s (v-t)/(\sigma^e \sqrt{v-s})}^{+\infty} d \phi (x) - K_s (v-t) \Phi \left( \frac{-K_s (v-t)}{\sigma^e \sqrt{v-s}} \right)
\]

\[
= \sigma^e \sqrt{v-s} \phi \left( \frac{K_s (v-t)}{\sigma^e \sqrt{v-s}} \right) - K_s (v-t) \Phi \left( \frac{-K_s (v-t)}{\sigma^e \sqrt{v-s}} \right)
\]

where \( \phi (x) = (1/\sqrt{2\pi}) \exp (-x^2/2) \) and \( \Phi (x) = \int_{-\infty}^x \phi (z) dz \). As \( K_t = k_c \), the expression for \( C_t (t, v) \) follows. Similarly, as \( - \int_s^v dR^e_u \) and \( \int_s^v dR^e_u \) have the same distribution under \( \mathbf{Q}_t \), the
equivalent expression for the put component is obtained by substituting \( k_p \) for \(-k_c\) using the fact that \( \phi(-x) = \phi(x) \). Taking derivatives with respect to \( v \) at \( s = t \) and taking Malliavin derivatives \( D^e_t \) gives the expressions for the Thetas and Deltas (use again \( \phi'(x) - x\phi(x) = 0 \) and \( D^e_t \) for the call or \( D^e_t \) for the put). Taking derivatives of the Greeks with respect to the horizon gives the formulas for Thetas of Deltas. Using the expressions for the Greeks and Corollary 1 gives the formula announced for the market price of risk.

Proof of Proposition 2: Timing option values and Thetas are bounded because \( \Phi(x) \in [0,1] \) and \( \phi(x) \leq 1/\sqrt{2\pi} \). The limits follow from the continuity of the expressions in Corollary 4.

Proof of Corollary 4: By construction of \( \theta^h \) in Proposition 1 and Corollary 3, \( Z_{t,v}^{h,e,1} \) is a local martingale. Furthermore, by the Markov inequality and the properties of the Greeks in \( \theta^h \),

\[
\mathbb{P}_t \left( \int_t^\tau \left( \theta^h_{t,v} \right)^2 dv \geq \lambda \right) \leq \frac{1}{\lambda} \int_t^\tau \mathbb{E}_t \left[ (\theta^h_{t,v})^2 \right] dv \leq \frac{k(t,\tau)}{\lambda}
\]

for some bounded deterministic function \( k(t,\tau) \) (the function is bounded as stated in Corollary 4 and Greeks are bounded). It follows that

\[
\mathbb{P}_t \left( \int_t^\tau \left( \theta^h_{t,v} \right)^2 dv \geq +\infty \right) = \lim_{\lambda \to +\infty} \mathbb{P}_t \left( \int_t^\tau \left( \theta^h_{t,v} \right)^2 dv \geq \lambda \right) = 0
\]

(because \( k(t,\tau)/\lambda \to 0 \)). This shows that \( Z_{t,v}^h \) is a strict martingale and that \( \mathbb{Q}_t \sim \mathbb{P}_t \). It follows that \( dR_v^e, dR_v^h \) are free from arbitrage opportunities.

Proof of Proposition 2: Standard arguments show that optimal wealth is \( X_t = y^{-1/R} \xi_t^{-1/R} \mathbb{E}_t \left[ \xi_{t,T}^0 \right] \), where \( \rho = 1 - 1/R \) and \( \xi_t = \xi_{0,t} \) with \( \xi_{t,v} \equiv \exp (\int_t^v r_s ds) Z_{t,v}^e \). As \( \pi'_t \sigma_{t,v} = D_t \log X_t \), it follows that \( \pi'_t \sigma_{t,v} = (1/R) \theta^0_{t,v} + D_t \log \mathbb{E}_t \left[ \xi_{t,T}^0 \right] \), where \( \theta^0_{t,v} \equiv [\theta^e, \theta^h] \) (see DGR (2003) for example). By assumption, \( \theta^0_{t,v} \) and \( r \) are deterministic, implying \( \mathbb{E}_t \left[ \xi_{t,T}^0 \right] \) deterministic. Thus, \( \pi'_t \sigma_{t,v} = (1/R) \theta^0_{t,v} \). Rearranging gives the expression announced.

Proof of Proposition 3: As in the Proof of Proposition 2, \( X_t = y^{-1/R} \xi_t^{-1/R} \mathbb{E}_t \left[ \xi_{t,T}^0 \right] \). First assume that \( T > t_1 \) and \( t > t_0 \) where \([t_0, t_1]\) is the first market timing interval. With \( t_0 = t \) and \( t_N = T \), and where \([t_j, t_{j+1}]\) denotes the \( j \)th timing window, it holds true that \( \mathbb{E}_t \left[ \xi_{t,T}^0 \right] = \mathbb{E}_t \left[ \xi_{t_1}^0 \prod_{j=1}^{N-1} \mathbb{E}_{t_j} \left[ \xi_{t_{j+1}}^0 \right] \right] \). As the conditioning in the product is at the starting point of each timing interval and \( \mathbb{E}_{t_j} \left[ \xi_{t_{j+1}}^0 \right] = k(j, t_j, t_{j+1}) \) for some constant \( k(j, t_j, t_{j+1}) \) (because \( \theta_{t_{j,v}} \) is conditionally independent from information available at \( t_j \)), it follows that

\[
\mathbb{E}_t \left[ \xi_{t,T}^0 \right] = \mathbb{E}_t \left[ \xi_{t_1}^0 \prod_{j=1}^{N-1} \mathbb{E}_{t_j} \left[ \xi_{t_{j+1}}^0 \right] \right] = \mathbb{E}_t \left[ \xi_{t_1}^0 \prod_{j=1}^{N-1} k(j, t_j, t_{j+1}) \right].
\]
Next, as \( r, \theta^e \) are constant and \( \theta^h_{t_0, \nu} \) is \( \mathcal{F}^e_t \)-adapted, it follows that

\[
e^{\rho(t_1-t)} \mathbb{E}_t [\xi^e_{t,t_1}] = \mathbb{E}_t \left[ (Z^e_{t,t_1})^\rho \mathbb{E}_t \left[ \left( Z^h_{t,t_1} \right)^\rho \mathcal{F}^e_{t_1} \right] \right] = \mathbb{E}_t \left[ (Z^e_{t,t_1})^\rho \exp \left( \frac{1}{2} \rho (\rho - 1) \int_t^{t_1} \left( \theta^h_{t_0, \nu} \right)^2 dv \right) \right].
\]

Therefore, using \( D_t \) as defined in (47) and \( \sum_{j=1}^{N-1} D_t \log k (j, t_j, t_{j+1}) = 0, \)

\[
D_t \log X_t = \frac{1}{R} \theta^h_{0, t} + D_t \log \mathbb{E}_t \left[ \left( Z^e_{t,t_1} \right)^\rho \exp \left( \frac{1}{2} \rho (\rho - 1) \int_t^{t_1} \left( \theta^h_{t_0, \nu} \right)^2 dv \right) \right].
\]

As \( D_t \) and \( \mathbb{E}_t \) commute, and \( D_t \log X_t = \pi^e_t \sigma_{t_0, t} \), it follows that

\[
\pi^e_t \sigma_{t_0, t} = \frac{1}{R} \theta^h_{0, t} + \rho (\rho - 1) \frac{\mathbb{E}_t \left[ V_{t,T} \int_t^{t_1} \theta^h_{t_0, \nu} D_t \theta^h_{t_0, \nu} dv \right]}{\mathbb{E}_t \left[ V_{t,T} \right]},
\]

where \( V_{t,T} \) is defined in (48). The expression for the optimal portfolio follows taking into account the fact that the same arguments apply if \( T < t_1 \).

**Proof of Proposition 4:** The proof follows Detemple and Rindisbacher (DR) (2005). Introduce the auxiliary state price density \( \xi_{t,T} \eta_{t,T} \equiv \xi_{t,T} \exp \left( - \int_t^T \nu_{-t} dW_{-t} - \frac{1}{2} \int_t^T \nu_{-t}^2 dv \right) \) and repeat the steps in the Proof of Proposition 2 to show \( \pi^h_t = (1/R) \left( (\theta^h + \nu_t) / \sigma^h \right) \). The auxiliary market price of risk \( \nu_t \) has to be chosen such that \( \pi^h_t > 0 \). As \( \theta^h = (\theta^h)^+ - (\theta^h)^- \) it follows that \( \nu_t = (\theta^h)^- \).

**Proof of Proposition 5:** Replicating the arguments in the Proof of Proposition 3 for \( \xi_{t,T} \eta_{t,T} \) defined in the Proof of Proposition 4, shows that, if \( \nu_t \) is \( \mathcal{F}^e \)-adapted, \( \pi^h_t = (1/R) \left( \theta^h_{t_0, t} + \nu_t \right) / \sigma^h \). As \( (\theta^h_{t_0, t})^- \) is indeed \( \mathcal{F}^e_t \)-adapted, it follows that \( D^2 \theta^h_{t_0, t} = 0 \). Therefore \( \nu_t = (\theta^h_{t_0, t})^- \) satisfies the backward equation for the shadow price derived in DR (2005). It follows that \( \theta^h_{t_0, t} \equiv \theta^h_{t_0, t} + \nu_t = (\theta^h_{t_0, t})^+ + (\theta^h_{t_0, t})^- = 0 \). Thus, \( Z^h_{t,v} \eta_{t,v} \) is a martingale. The expression announced then follow by the same reasoning as in the proof of Proposition 3 replacing \( \theta^h_{t_0, t} \) by \( \theta^h_{t_0, t} \).

**Proof of Proposition 6:** The optimal portfolio is derived using the same arguments as in the proofs of Propositions 2 and 4, but with the new expressions for \( \sigma_{t_0, t} \) and \( \theta_t = \left[ \theta^h, \theta^f, \theta^h_{t_0, t} \right] \).

**Proof of Proposition 7:** The optimal portfolio is derived using the arguments in the proofs of Propositions 3 and 5, but with the new expressions for \( \sigma_{t_0, t} \) and \( \theta_t = \left[ \theta^h, \theta^f, \theta^h_{t_0, t} \right] \). The proof of Proposition 5 shows that \( \mathbb{E}_t \left[ \xi^e_{t,T} \right] = \mathbb{E}_t \left[ V^e_{t,T} \right] \exp \left( \frac{1}{2} \rho (\rho - 1) \left\| \theta^f \right\|^2 (t_1 - t) - \frac{1}{2} \rho (\theta^e)^2 (t_1 - t) \right) \) (because \( W^f \) and \( W_e \) are independent and \( \theta^f \) is constant). Taking Malliavin derivatives shows that the hedging demand does not depend on \( W^f \). The expression announced follows.

**Proof of Proposition 8:** Let \( \lambda^f \) be the Lagrange multiplier associated with the no-holding constraint on additional risk factors. The optimal unconstrained portfolio (equities and factors) is

\[
\frac{1}{R} \left[ \frac{\theta^e}{\sigma^e} - \frac{\Sigma^e_{ff} (\Sigma^f_{ff})^{-1}}{\sigma^e} (\theta^f + \lambda^f) - \kappa^e \theta^h_{t_0, \nu} \right] = \frac{\theta^h_{t_0, \nu}}{\sigma^h} \left[ (\Sigma^f_{ff})^{-1} (\theta^f + \lambda^f) - \delta \frac{\theta^h_{t_0, \nu}}{\sigma^h} \right].
\]
The constrained portfolio is such that the second line is null (because factors are not investable assets). This gives \( \lambda^f = \Sigma_{ff} \delta_{h,TM,c}^t - \theta^f \). Substituting in the first line gives the mean-variance market weight for equity holdings. Moreover, \( \lambda^f \) is deterministic because \( \theta_{h,TM,c}^t \) and \( \sigma^h \) are non-stochastic. Thus, the constraint does not introduce an additional hedging motive. Similarly, given the triangular volatility structure, the hedge fund weight is unaffected by \( W^e \)-risk. It therefore remains the same as in the absence of additional risk factors.

**Proof of Proposition 9:** As in the Proof of Proposition 8, the individual price of risk associated with the no-holding constraint is \( \theta^f + \lambda^f = \Sigma_{ff} \delta_{h,HM,c}^t - \theta^f \). Substituting in the individual state price density and using the same conditioning arguments as in the proofs of Propositions 7 and 5 gives the hedging portfolio component. Given the triangular volatility structure, the hedge fund weight is not affected by \( W^e \)-risk and therefore does not change in the presence of nontraded factors.

**Proof of Proposition 10:** The definition of the certainty equivalent \( \hat{x} \) gives

\[
\frac{x^{1-R}}{1-R} = \frac{x^{1-R}}{1-R} \left( E \left[ \xi_T^\rho \right] \right)^R \iff \hat{x} = x \left( E \left[ \xi_T^\rho \right] \right)^{R/(1-R)}.
\]

Simple calculations show that

\[
E \left[ \xi_T^\rho \right] = \exp \left( -\rho r T + \frac{1}{2} \rho (\rho - 1)(\theta^e)^2 T \right)
\]

\[
\times E \left[ \exp \left( -\rho^e W_T^e - \frac{1}{2} (\rho^e)^2 T + \frac{1}{2} \rho (\rho - 1) \int_0^T (\theta_s^h)^2 ds \right) \right]
\]

with hedge fund investment \( \theta^h \neq 0 \) and \( E \left[ \xi_T^\rho \right] = \exp \left( -\rho r T + \frac{1}{2} \rho (\rho - 1)(\theta^e)^2 T \right) \) without \( \theta^h = 0 \). The ratio of certainty-equivalents for HM follows. For TM strategies the market price of hedge fund risk is constant, implying \( E \left[ \xi_T^\rho \right] = \exp \left( -\rho r T + \frac{1}{2} \rho (\rho - 1)(\theta^e)^2 T \right) \times \exp \left( \frac{1}{2} \rho (\rho - 1) \int_0^T (\theta_s^h)^2 ds \right) \). The result announced follows.

**Proof of Equations (67-69):** Let \( t_{i+1} - t_i = \Delta \). The TM regression gives

\[
\log \left( \frac{1 + r_{t_{i+1}}^h}{1 + r_{t_{i+1}}^f} \right) = \log S_{t_{i+1}}^h - \log S_{t_i}^h - (\log b_{t_{i+1}} - \log b_{t_i})
\]

\[
= \int_{t_i}^{t_{i+1}} dR_v^h - \frac{1}{2} \int_{t_i}^{t_{i+1}} \left( (\sigma_{h,v}^e)^2 + (\sigma_{h,p}^h)^2 \right) dv
\]

\[
= \left( \alpha - \frac{1}{2} (\sigma_{h,p}^h)^2 \right) \Delta + \beta \int_{t_i}^{t_{i+1}} dR_v^e + \gamma^{TM} \left( \int_{t_i}^{t_{i+1}} dR_v^e \right)^2 + \sigma_{h,p}^h \left( \tilde{W}_{t_{i+1}}^p - \tilde{W}_{t_i}^p \right)
\]

\[
- \frac{1}{2} \int_{t_i}^{t_{i+1}} (\sigma_{h,v}^e)^2 dv.
\]
By Ito’s lemma, \( \int_{t_i}^{t_{i+1}} dR_v^e = \log \left( \frac{1+r_{t_{i+1}}^e}{1+r_{t_{i}}^j} \right) + \frac{1}{2} (\sigma^e)^2 \Delta \), so that

\[
\left( \int_{t_i}^{t_{i+1}} dR_v^e \right)^2 = \left( \log \left( \frac{1+r_{t_{i+1}}^e}{1+r_{t_{i}}^j} \right) \right)^2 + \frac{1}{4} (\sigma^e)^4 \Delta^2 + \log \left( \frac{1+r_{t_{i+1}}^e}{1+r_{t_{i}}^j} \right) (\sigma^e)^2 \Delta.
\]

Collecting terms leads to (67)-(68). To establish (69), note that \( \sigma_{t_i,v}^{h,e} = \sigma^e \left( \beta + 2\gamma^TM \int_{t_i}^{t_i} dR_s^e \right) \), so that

\[
\frac{1}{(\sigma^e)^2} \int_{t_i}^{t_{i+1}} \left( \sigma_{t_i,v}^{h,e} \right)^2 dv = \beta^2 \Delta + 4\beta \gamma^TM \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_i} dR_v^e dv + 4 \left( \gamma^TM \right)^2 \int_{t_i}^{t_{i+1}} \left( \int_{t_i}^{t_i} dR_v^e \right)^2 dv
= \beta^2 \Delta + 4\beta \gamma^TM \int_{t_i}^{t_{i+1}} \log \left( \frac{1+r_{t_{i+1}}^e}{1+r_{t_{i}}^j} \right) + \frac{1}{2} (\sigma^e)^2 (v-t_i) dv
+ 4 \left( \gamma^TM \right)^2 \int_{t_i}^{t_{i+1}} \left( \log \left( \frac{1+r_{t_{i+1}}^e}{1+r_{t_{i}}^j} \right) + \frac{1}{2} (\sigma^e)^2 (v-t_i) \right)^2 dv
= \beta^2 \Delta + 4\beta \gamma^TM \int_{t_i}^{t_{i+1}} \log \left( \frac{1+r_{t_{i+1}}^e}{1+r_{t_{i}}^j} \right) dv + 2\beta \gamma^TM (\sigma^e)^2 \int_{t_i}^{t_{i+1}} (v-t_i) dv
+ 4 \left( \gamma^TM \right)^2 \int_{t_i}^{t_{i+1}} \log \left( \frac{1+r_{t_{i+1}}^e}{1+r_{t_{i}}^j} \right)^2 dv
+ 4 \left( \gamma^TM \sigma^e \right)^2 \int_{t_i}^{t_{i+1}} \log \left( \frac{1+r_{t_{i+1}}^e}{1+r_{t_{i}}^j} \right) (v-t_i) dv
+ \left( \gamma^TM \right)^2 (\sigma^e)^4 \int_{t_i}^{t_{i+1}} (v-t_i)^2 dv.
\]

Using \( \int_{t_i}^{t_{i+1}} (v-t_i) dv = \frac{1}{2} \Delta^2 \) and \( \int_{t_i}^{t_{i+1}} (v-t_i)^2 dv = \frac{1}{3} \Delta^3 \) leads to (69). \[\blacksquare\]
References


Table 1: **Return Statistics on HFRI Equally-Weighted Indices**

This table shows the summary statistics of (annualized) percentage returns for equally-weighted HFRI indices of the main hedge fund strategies from January 1996 till August 2009 (164 observations). Panel a reports the statistics for raw returns, Panel b for unsmoothed returns, and Panel c for the returns on the market and on risk factors. SR denotes the Sharpe Ratio, Corr. the correlation with the market.

### Panel a: Raw Returns

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### Panel b: Unsmoothed Returns

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### Panel c: Market and Risk Factor Returns

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Table 2: HM and TM Parameter Estimates for Equally-Weighted HFRI Indices
Panels a and b report the point estimates and the t-statistics for the parameters of the Henriksson-Merton (HM), panels c and d those of the Treynor-Mazuy (TM) regression models described in Sections 5.1.1 and 5.1. Panels a and c are for reported returns, panels b and d for unsmoothed returns (see Section 6.1).

| Panel a: HM Estimation Results for Reported Returns without Risk Factors |
|-----------------------------|-------------------|-------------|-------------|-------------|-------------|-------------|-------------|
|                              | α     | $t(\alpha)$ | β     | $t(\beta)$ | γ     | $t(\gamma)$ | $\sigma^{h,p}$ | adj. $R^2$ |
| Equity Hedge-Total           | 0.0788| 2.7227      | 0.4668| 10.4717     | 0.0498| 0.1716       | 0.0541       | 0.2963      |
| Equity Hedge-Short Bias      | 0.1763| 2.6930      | -0.7061| -8.7735    | -1.5543| -2.4316      | 0.1190       | 0.3587      |
| Event Driven-Total           | 0.1285| 5.7084      | 0.4118| 16.8570     | -0.6591| -2.9632      | 0.0431       | 0.3492      |
| Macro-Total                  | 0.0466| 1.6356      | 0.0809| 2.0731      | 0.4835| 1.8273       | 0.0593       | 0.8261      |
| Macro-Systematic Diversified | 0.0166| 0.5517      | 0.0594| 1.9403      | 1.2711| 5.1894       | 0.0662       | 0.7047      |
| Relative Value-Total         | 0.1173| 6.2762      | 0.2470| 15.7769     | -0.6605| -3.9728      | 0.0356       | 0.5814      |
| Emerging Markets-Total       | 0.1661| 3.2510      | 0.7990| 15.0545     | -1.3243| -2.7997      | 0.1038       | 0.4681      |
| Funds of Funds Composite     | 0.0841| 3.3409      | 0.3119| 10.7251     | -0.5098| -2.2933      | 0.0474       | 0.5314      |

| Panel b: HM Estimation Results for Unsmoothed Returns without Risk Factors |
|-----------------------------|-------------------|-------------|-------------|-------------|-------------|-------------|-------------|
|                              | α     | $t(\alpha)$ | β     | $t(\beta)$ | γ     | $t(\gamma)$ | $\sigma^{h,p}$ | adj. $R^2$ |
| Equity Hedge-Total           | 0.1042| 2.6253      | 0.6132| 14.5680     | -0.4741| -1.1931      | 0.0732       | 0.3579      |
| Equity Hedge-Short Bias      | 0.1292| 1.7690      | 1.1164| 14.6794     | -1.1403| -1.5036      | 0.1564       | 0.4792      |
| Event Driven-Total           | 0.0646| 2.1431      | 0.3263| 9.1072      | -0.3618| -1.2095      | 0.0603       | 0.6000      |
| Macro-Total                  | 0.0106| 0.2796      | 0.0841| 1.6713      | 0.9287| 2.5984       | 0.0816       | 0.8124      |
| Macro-Systematic Diversified | -0.0239| -0.6395   | 0.0596| 1.4925      | 1.7946| 5.6767       | 0.0851       | 0.6932      |
| Relative Value-Total         | 0.1111| 3.0692      | 0.3839| 11.5277     | -0.6801| -2.1094      | 0.0642       | 0.6785      |
| Emerging Markets-Total       | -0.0248| -0.5866   | 0.7384| 5.6690      | 1.1138| 1.1719       | 0.1520       | 0.4913      |
| Funds of Funds Composite     | 0.0437| 3.3373      | 0.5339| 14.6925     | 0.2822| 0.8851       | 0.0664       | 0.3013      |

| Panel c: TM Estimation Results for Reported Returns without Risk Factors |
|-----------------------------|-------------------|-------------|-------------|-------------|-------------|-------------|-------------|
|                              | α     | $t(\alpha)$ | β     | $t(\beta)$ | γ     | $t(\gamma)$ | $\sigma^{h,p}$ | adj. $R^2$ |
| Equity Hedge-Total           | 0.0492| 0.7980      | 0.4698| 4.9398      | -0.0743| -0.0729      | 0.0536       | 0.2963      |
| Equity Hedge-Short Bias      | 0.1644| 0.5954      | -0.9755| -6.4397    | -1.8596| -0.8252      | 0.1184       | 0.3587      |
| Event Driven-Total           | 0.0946| 1.3699      | 0.2074| 3.9846      | -0.9033| -1.1290      | 0.0421       | 0.3492      |
| Macro-Total                  | 0.0384| 0.6728      | 0.1584| 1.4948      | 0.5225| 0.4600       | 0.0597       | 0.8261      |
| Macro-Systematic Diversified | 0.0853| 1.0430      | 0.2818| 2.4867      | 1.7922| 1.4756       | 0.0638       | 0.7047      |
| Relative Value-Total         | 0.0964| 1.6616      | 0.1252| 2.1307      | -1.0636| -1.6891      | 0.0331       | 0.5814      |
| Emerging Markets-Total       | 0.1619| 0.6960      | 0.5747| 3.1200      | -1.7402| -0.8813      | 0.1038       | 0.4681      |
| Funds of Funds Composite     | 0.0490| 0.6933      | 0.2227| 2.6868      | -0.7075| -0.7968      | 0.0467       | 0.5314      |

| Panel d: TM Estimation Results for Unsmoothed Returns without Risk Factors |
|-----------------------------|-------------------|-------------|-------------|-------------|-------------|-------------|-------------|
|                              | α     | $t(\alpha)$ | β     | $t(\beta)$ | γ     | $t(\gamma)$ | $\sigma^{h,p}$ | adj. $R^2$ |
| Equity Hedge-Total           | 0.0753| 0.6557      | 0.5254| 4.0518      | -0.7984| -0.5745      | 0.0730       | 0.3647      |
| Equity Hedge-Short Bias      | 0.1362| 0.3909      | 0.9158| 3.2890      | -1.7123| -0.5737      | 0.1569       | 0.4886      |
| Event Driven-Total           | 0.0347| 0.3995      | 0.2589| 2.4342      | -0.6047| -0.5305      | 0.0599       | 0.5974      |
| Macro-Total                  | 0.0360| 0.4547      | 0.2329| 1.5983      | 0.9936| 0.6362       | 0.0821       | 0.8234      |
| Macro-Systematic Diversified | 0.1233| 0.8772      | 0.3710| 2.5482      | 2.4757| 1.5864       | 0.0820       | 0.6470      |
| Relative Value-Total         | 0.1004| 0.8677      | 0.2550| 2.2856      | -1.1887| -0.9940      | 0.0628       | 0.5809      |
| Emerging Markets-Total       | 0.0024| 0.0140      | 0.9271| 3.4157      | 1.4591| 0.5015       | 0.1529       | 0.5034      |
| Funds of Funds Composite     | 0.0215| 0.3095      | 0.5751| 4.8375      | 0.2030| 0.1593       | 0.0670       | 0.3155      |
Table 3: HM Parameter Estimates for Equally-Weighted HFRI indices with Additional Risk Factors

Panel a reports the point estimates and the t-statistics for the parameters of the Henriksson-Merton (HM) regression model described in section 5.1.1 with seven additional risk factors. Panel b reports the factor loadings of the four factors considered in Lo and Hasanhodzic (2006) (Bond, Credit, Commodity and FX index) and the three Fama-French-Carhart factors (Size (SMB), Book-to-Market (HML), and Momentum).

### Panel a: Estimation Results for Reported Returns with Risk Factors

<table>
<thead>
<tr>
<th></th>
<th>(\alpha)</th>
<th>(t(\alpha))</th>
<th>(\beta)</th>
<th>(t(\beta))</th>
<th>(\gamma)</th>
<th>(t(\gamma))</th>
<th>(\sigma^{h,p})</th>
<th>adj. (R^2)</th>
</tr>
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<tbody>
<tr>
<td>Equity Hedge-Total</td>
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<td>0.3772</td>
<td>8.0283</td>
<td>0.2997</td>
<td>1.2509</td>
<td>0.0414</td>
<td>0.1711</td>
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<td>Equity Hedge-Short Bias</td>
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<td>3.7477</td>
<td>-0.6039</td>
<td>-6.9703</td>
<td>-1.2801</td>
<td>-2.8453</td>
<td>0.0741</td>
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<td>Event Driven-Total</td>
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<td>Macro-Total</td>
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<td>Relative Value-Total</td>
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<td>0.3676</td>
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<td>0.0979</td>
<td>0.4157</td>
</tr>
<tr>
<td>Funds of Funds Composite</td>
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<td>-0.2080</td>
<td>-1.1024</td>
<td>0.0391</td>
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### Panel b: Factor Loadings (Lo-Hasanhodzic and Fama-French-Carhart Risk Factors)

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<th>(\delta_{Bond})</th>
<th>(t(\delta_{Bond}))</th>
<th>(\delta_{Credit})</th>
<th>(t(\delta_{Credit}))</th>
<th>(\delta_{Comdt})</th>
<th>(t(\delta_{Comdt}))</th>
<th>(\delta_{FX})</th>
<th>(t(\delta_{FX}))</th>
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<td>3.3998</td>
<td>-0.0064</td>
<td>-0.1269</td>
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<tr>
<td>Equity Hedge-Short Bias</td>
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<td>0.0113</td>
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<tr>
<td>Event Driven-Total</td>
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<td>0.8972</td>
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<td>-0.5590</td>
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<td>Macro-Total</td>
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<tr>
<td>Macro-Systematic Diversified</td>
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<td>0.1272</td>
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<td>Relative Value-Total</td>
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<td>Emerging Markets-Total</td>
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<td>0.1407</td>
<td>0.3813</td>
<td>1.6721</td>
<td>0.0690</td>
<td>1.5694</td>
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</tr>
<tr>
<td>Funds of Funds Composite</td>
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<td>0.0492</td>
<td>3.0243</td>
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<td>-2.9349</td>
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Fama-French-Carhart Risk Factors

<table>
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<th>(\delta_{SMB})</th>
<th>(t(\delta_{SMB}))</th>
<th>(\delta_{HML})</th>
<th>(t(\delta_{HML}))</th>
<th>(\delta_{MOM})</th>
<th>(t(\delta_{MOM}))</th>
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<tbody>
<tr>
<td>Equity Hedge-Total</td>
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<tr>
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<tr>
<td>Event Driven-Total</td>
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<td>5.1444</td>
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<td>0.0212</td>
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<tr>
<td>Macro-Total</td>
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<tr>
<td>Macro-Systematic Diversified</td>
<td>0.0207</td>
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<td>Relative Value-Total</td>
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</table>
Table 4: HM and TM Parameter Estimates for Equally-Weighted HFRI Indices with Monthly Reported Returns

This table reports the point estimates and the t-statistics for the parameters of the Henriksson and Merton (HM) and Treynor and Mazui (TM) linear regression models at the monthly frequency, in Panel a and b respectively.

Panel a: HM Estimation Results for Monthly Reported Returns without Risk Factors

<table>
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<tr>
<th></th>
<th>α</th>
<th>t(α)</th>
<th>β</th>
<th>t(β)</th>
<th>γ</th>
<th>t(γ)</th>
<th>σ^h,p</th>
<th>adj. R^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity Hedge-Total</td>
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<tr>
<td>Equity Hedge-Short Bias</td>
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<tr>
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<td>-0.1597</td>
<td>-2.1581</td>
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</table>

Panel b: TM Estimation Results for Monthly Reported Returns without Risk Factors

<table>
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<tr>
<th></th>
<th>α</th>
<th>t(α)</th>
<th>β</th>
<th>t(β)</th>
<th>γ</th>
<th>t(γ)</th>
<th>σ^h,p</th>
<th>adj. R^2</th>
</tr>
</thead>
<tbody>
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<td>Equity Hedge-Total</td>
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Figure 1: Excess Return Scenarios, Correlation and Mean-Variance Market Weights
This figure shows time-varying correlations and mean-variance market weights for four different cumulative return scenarios for HM (left panels) and TM (right panels) market timing models. Portfolio weights for hedge funds are subject to no-short sales constraints. Parameters are those estimated for the Macro-Systematic Diversified style index in Panel a in Table 2 for HM and Panel c of Table 2 for TM. Relative risk aversion is $R = 5$. 

Cumulative Excess Return Scenarios

Correlation HF and Market: HM

Mean-Variance Market Weight: HM

Correlation HF and Market: TM

Mean-Variance Market Weight: TM
Figure 2: Portfolio Weights Up-Up Market
This figure shows the cumulative returns, the market volatility coefficient of hedge fund returns, the market price of hedge fund risk, the market and hedge fund weights with and without short sales constraints. The dotted line in the last two left panels is the mean-variance component and the bold line the total portfolio weight. Their difference is the dynamic market timing hedging portfolio. Parameter estimates are those in Panel a Table 2. Relative risk aversion is $R = 5$. 

Cumulative Excess Returns

Volatility $\sigma^{h,e}_{1,t}$

MPR $\theta^{h}_{1,t}$

Market Weight

Hedge Fund Weight

Market Weight, Short-sales Constraint

Hedge Fund Weight, Short-sales Constraint
Figure 3: Portfolio Weights Down-Down Market
This figure shows the cumulative returns, the market volatility coefficient of hedge fund returns, the market price of hedge fund risk, the market and hedge fund weights with and without short sales constraints. The dotted line in the last two left panels is the mean-variance component and the bold line the total portfolio weight. Their difference is the dynamic market timing hedging portfolio. Parameter estimates are those in Panel a Table 2. Relative risk aversion is $R = 5$. 

Cumulative Excess Returns

Volatility $\sigma^{h,t}_{\text{MPR}}$

MPR $\theta^{h}_{\text{MPR}}$

Market Weight

Hedge Fund Weight

Market Weight, Short-sales Constraint

Hedge Fund Weight, Short-sales Constraint
Figure 4: Portfolio Weights Down-Up Market

This figure shows the cumulative returns, the market volatility coefficient of hedge fund returns, the market price of hedge fund risk, the market and hedge fund weights with and without short sales constraints. The dotted line in the last two left panels is the mean-variance component and the bold line the total portfolio weight. Their difference is the dynamic market timing hedging portfolio. Parameter estimates are those in Panel a Table 2. Relative risk aversion is $R = 5$. 

Cumulative Excess Returns

Volatility $\sigma_{h,t}^{b,c}$

MPR $\theta_{h,t}^b$

Market Weight

Hedge Fund Weight

Market Weight, Short-sales Constraint

Hedge Fund Weight, Short-sales Constraint
Figure 5: **Portfolio Weights Up-Down Market**

This figure shows the cumulative returns, the market volatility coefficient of hedge fund returns, the market price of hedge fund risk, the market and hedge fund weights with and without short sales constraints. The dotted line in the last two left panels is the mean-variance component and the bold line the total portfolio weight. Their difference is the dynamic market timing hedging portfolio. Parameter estimates are those in Panel a Table 2. Relative risk aversion is $R = 5$. 

Cumulative Excess Returns

Volatility $\sigma_{t_0,t}^{h,e}$  

MPR $\theta_{t_0,t}^{h}$

Market Weight

Hedge Fund Weight

Market Weight, Short-sales Constraint

Hedge Fund Weight, Short-sales Constraint
This figure shows portfolio weights with additional risk factors. Rows correspond to the scenarios “up-up”, “down-down”, “down-up”, and “up-down”. The left panels show the total portfolio weights in the market (solid line), the Credit factor (dashed line), the HML portfolio (dotted line) and the Macro-Systematic Diversified hedge fund style index (dash-dotted line) with no-short sales constraint in hedge fund. The middle (right) panels separate the total market weight and the mean-variance weight if risk factors (Credit, HML) are traded (non-traded) portfolios. The dashed lines in the middle and left panels are the mean variance portfolio weights and the solid lines the total portfolio weights. Parameters are estimated from reported returns with a Credit and HML factor. Parameter estimates are $\alpha = 0$, $\beta = 0.1197$, $\gamma = 1.0371$, $\sigma^{h,p} = 0.0593$, $\delta_{Credit} = -0.4165$ and $\delta_{HML} = -0.1585$. Relative risk aversion is $R = 5$. 

$$\text{Total Weights} \quad \text{Market Weight (Traded Factors)} \quad \text{Market Weight (Non-traded Factors)}$$

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure6.pdf}
\end{figure}
Figure 7: Portfolio Weights Up-Down Market with Credit and HML Factors

This figure shows the portfolio weights with Credit and HML factors for the “up-down” market scenario, in the presence of a short sales constraint on hedge fund investments. Rows 2 and 3 show the portfolio with traded factors, row 4 with non-traded factors. Parameters are those reported in Figure 6. Relative risk aversion is $R=5$.

Cumulative Excess Returns

Market Weight

Hedge Fund Weight

Credit Weight

HML Weight

Market Weight, Non-traded Factors

Hedge Fund Weight, Non-traded Factors

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Relative certainty equivalents for short-sale constrained HM hedge fund strategies are presented in the left panel. The right panel reports the cost of short sales constraints in terms of differences of relative certainty equivalents with and without short sales constraints. Parameter values are those for the Systematic-Diversified style in Panel a Table 2. Estimated market coefficients are $\sigma^e = 0.2$, $\theta^e = 0.3892$. Risk aversion varies from 2 to 5. The timing window is one month. Results are reported for investment horizons from 1 months to 3 years. It is assumed that timing ability persists for each month within the investment horizon. Monthly relative certainty equivalents under the short sale constraint are 1.0091, 1.0061, 1.0045 and 1.0036 for $R = 2, 3, 4, 5$. 